POINTS SETS AND ALLIED CREMONA GROUPS*

(PART I)

BY

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Introduction

The theory of invariants in the binary domain admits of generalization in two directions. If the emphasis be placed on the binary *form* of order n, the natural extension, and the one hitherto always considered, is to forms in a larger number of variables. If however the emphasis be laid on the set of n points, we are led to consider the invariant or projective properties of sets, P_n^k, of n points in S_k. Though it would seem that, since the P_n^k can be determined by a degenerate form of class k in S_k, the latter extension is contained in the former as a degenerate case, yet this is not true. The divergence of the two generalizations is due to the difference in their domains of rationality which in the first case is the domain of the coefficients of the form and in the second case is the domain of symmetric functions of the coördinates of the points of P_n^k. For example we shall find that P_6^2 has an invariant of degree two while the lowest degree of an invariant of the ternary sextic is three.

One object of this paper is to begin the formulation of a theory of invariants of the point set P_n^k. In § 3 and § 5 invariants of P_n^k are defined and complete systems for P_6^1 and P_6^2 are derived from this definition. The first two paragraphs are devoted to the *association* of a set P_n^k with a set Q_n^{n-k-2}. Sets which are thus associated have the same absolute invariants. Much of the material collected here would seem to be useful for other purposes. A study of the earlier cases indicates strongly that the lines followed in §§ 1, 2, and 9 will lead to important results in connection with the theta functions.

The first purpose, outlined above, is interrupted, perhaps rather rudely, in order to study an application of P_n^k to the theory of equations. The point set defines a Cremona group of order $n!$, $G_{n!}$, in a space $\Sigma_{k(n-k-2)}$. The $G_{n!}$ is a generalization of Moore's‡ "Cross-ratio Group." It is discussed

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‡ E. H. Moore, American Journal of Mathematics, vol. 22 (1900), p. 279.

in §§ 6, 7 while the form-problems of the group for sets P_5 and P_6 are considered in §§ 8, 10 in connection with related papers of the writer.*

If $n > k + 3$, the $G_{n!}$ is a subgroup of a larger group in $\Sigma_{k(n-k-2)}$ which also is defined by P_n^k. This extended group is in general infinite and discontinuous though in certain very interesting cases it is finite. As n and k increase we obtain a doubly infinite series of extended groups which will be discussed in Part II of this paper.

For the particular set P_6^2 this extended group is finite and of order 51840. The rôle played by this group in the problem of the determination of the right lines on a cubic surface will be considered in Part III. For this purpose the matters developed in § 4 and § 5 of the present account will be of importance.

1. Associated and self-associated point sets

Let the set P_n^k of n points in S_k be given by the equations

$$(up_1) = 0, \qquad (up_2) = 0, \qquad \cdots, \qquad (up_n) = 0.$$

These equations are connected by $n - k - 1$ linear relations since in S_k any $k + 2$ points must be linearly dependent. If we take these relations to be

$$q_{1i}(up_1) + q_{2i}(up_2) + \cdots + q_{ni}(up_n) \equiv 0 \qquad (i = 1, \cdots, n - k - 1), \tag{1}$$

we can, by multiplying them respectively by v_i and adding, express them by the single identity in u and v

$$(vq_1)(up_1) + (vq_2)(up_2) + \cdots + (vq_n)(up_n) \equiv 0. \tag{2}$$

Thus the identities (1) lead to a set, Q_n^{n-k-2}, of n points in S_{n-k-2}. The identities may however be replaced by any $n - k - 1$ independent linear combinations so that the points (vq) in S_{n-k-2} are determined only to within linear transformation. Moreover from the symmetry of (2) in points p and points q we can infer at once that the same projectively defined set Q_n^{n-k-2} is determined by any set projectively equivalent to P_n^k.

(3) *The class of projectively equivalent point sets P_n^k defines in (2) an "associated" class of projectively equivalent sets Q_n^{n-k-2} such that the relation between the two is mutual.*

Two point sets—one from each class—will be called *associated sets.* Unless the context clearly indicates the contrary it will be understood that a point set is any one of its class.

* *Cremona groups and the solution of equations,* vol. 9 (1908), p. 396; *The cross-ratio group and the solution of the sextic,* vol. 12 (1911), p. 311; *Finite geometry and theta functions,* vol. 14 (1913), p. 241; all in these Transactions. These papers are cited hereafter as $C1$, $C2$, $C3$ respectively.

If a P_4^1 is associated with a Q_4^1 the identity (2) reads

$$(4) \qquad (vq_1)(up_1) + \cdots + (vq_4)(up_4) \equiv 0.$$

Due to this identity all four pairs $(vq_i) \cdot (up_i)$ are apolar to the form $(ay)(bx)$ apolar to any three, i. e., the four pairs correspond in the binary projectivity, $(ay)(bx) = 0$, and the two sets are projective.

If a set P_n^1 is associated with Q_n^{n-3}, and if we utilize any $n - 4$ of the points q, say $q_5, \cdots, q_n$, in order to form the contragredient symbol

$$(q_5, q_6, \cdots, q_n, y, y')$$

and then operate on (2) (y' operating on v), we obtain the new identity

$$(q_5, \cdots, q_n, y, q_1)(up_1) + (q_5, \cdots, q_n, y, q_2)(up_2) + \cdots$$
$$+ (q_5, \cdots, q_n, y, q_4)(up_4) \equiv 0.$$

Comparing this with (4) we see that the four S_{n-4}'s of the pencil on $q_5, \cdots, q_n$ which pass through $q_1, \cdots, q_4$ respectively are projective to $p_1, \cdots, p_4$ in S_1. Thus the set Q_n^{n-3} in S_{n-3} is projective on the rational norm-curve of order $n - 3$ through it to the set P_n^1 in S_1. This property is characteristic since it requires that two sets Q_n^{n-3} and $Q_n'^{n-3}$ each associated with P_n^1 be projectively equivalent. Let us apply it to the case of the associated P_n^k and Q_n^{n-k-2}. In S_{n-k-2} let any $n - k - 3$ of the points q be isolated, say $q_1, \cdots, q_{n-k-3}$. On these points there is a pencil of S_{n-k-3}'s, $k + 3$ of which are determined by the points $q_{n-k-2}, \cdots, q_n$. By operating on (2) with $(q_1, \cdots, q_{n-k-3}, y, y')$ (y' operating on v), we obtain

$$(q_1, \cdots, q_{n-k-3}, y, q_{n-k-2})(up_{n-k-2})$$
$$+ (q_1, \cdots, q_{n-k-3}, y, q_{n-k-1})(up_{n-k-1}) + \cdots$$
$$+ (q_1, \cdots, q_{n-k-3}, y, q_n)(up_n) \equiv 0.$$

Thus the $k + 3$ members of the pencil (a $\bar{Q}_{k+3}^1$) are associated with the set $p_{n-k-2}, \cdots, p_n$ (a $\bar{P}_{k+3}^k$). Hence

(5) *If the sets P_n^k and Q_n^{n-k-2} are associated any group of $k + 3$ points p are, when taken on the rational norm-curve through them, projective to the members of the pencil of S_{n-k-3}'s in S_{n-k-2} which are determined by the complementary set of $n - k - 3$ points q and each in turn of the corresponding $k + 3$ points q.*

Naturally the theorem (5) is still true if P_n^k and Q_n^{n-k-2} be interchanged and k be replaced by $n - k - 2$. The property (5) is characteristic of associated sets. For if Q_n^{n-k-2} be given and $k + 2$ points of P_n^k be fixed in S_k, the $n - k - 2$ norm-curves on these $k + 2$ points each carrying a further point of P_n^k are uniquely determined, and the position of the further point also is uniquely determined by the projective conditions on the parameters.

Another property of associated sets is the following:

(6) *If by projection from* r *of the points* q, *say* $q_{n-r+1}, \cdots, q_n$, *and section by an* $S_{n-k-2-r}$ *there is obtained from the* $n - r$ *remaining points* q *a set* $Q_{n-r}^{n-k-2-r}$, *this set is associated with the set* P_{n-r}^k *formed by the points* $p_1, \cdots, p_{n-r}$ *of* P_n^k.

This is proved by operating on the identity (2) with the symbol

$$(q_{n-r+1}, \cdots, q_n, y, y_1, \cdots, y_{n-k-2-r}),$$

y operating on v. In this symbol $y_1, \cdots, y_{n-k-2-r}$ can be regarded as a coördinate v' in $S_{n-k-2-r}$.

The identity expresses that the matrix M_p of n columns and $k + 1$ rows formed from the coördinates of P_n^k and the matrix M_q of n columns and $n - k - 1$ rows formed from the coördinates of Q_n^{n-k-2} are conjugate. The theorem of Grassman that

(7) *Complementary determinants of the matrices* M_p *and* M_q *are proportional* can be proved at once by means of the identity. For if we operate on (2) with the symbol $(q_1, q_2, \cdots, q_{n-k-2}, y)(x, p_{n-k+1}, \cdots, p_n)$, the y operating on v and the x on u, we get

$$(q_1, \cdots, q_{n-k-2}, q_{n-k-1})(p_{n-k-1}, p_{n-k+1}, \cdots, p_n) + (q_1, \cdots, q_{n-k-2}, q_{n-k})(p_{n-k}, p_{n-k+1}, \cdots, p_n) = 0.$$

If we set

$$(q_1, q_2, \cdots, q_{n-k-2}, q_{n-k-1}) = \lambda(p_{n-k}, p_{n-k+1}, \cdots, p_n),$$

then

$$(q_1, q_2, \cdots, q_{n-k-2}, q_{n-k}) = -\lambda(p_{n-k-1}, p_{n-k+1}, \cdots, p_n).$$

Other equalities can be deduced from these as the second is deduced from the first until the theorem is completely established.

Let us define the number

(8) $$N_{n,k} = 2k + 2 - n$$

to be the *norm* of P_n^k. For associated sets,

(9) $$N_{n,k} + N_{n,n-k-2} = 0.$$

In many respects those sets whose norm is zero have the simplest projective properties. Given then the set P_{2k+2}^k let us obtain a construction for its associated Q_{2k+2}^k. Suppose that the two sets are in the same S_k so that the first $k + 2$ points of the one set coincide respectively with the corresponding first $k + 2$ points of the other and form in S_k a *base* (i. e., $k + 2$ points such that no $k + 1$ lie in an S_{k-1}). A manifold containing the points of the base will be called a *basic* manifold. We shall need the following theorems:*

(*a*) *Basic rational norm-curves,* R^k, *which touch an* S_{k-1}, α, *touch at points of a quadric,* Q_α, *in* α.

* For these three theorems and further references, see Conner, *Basic systems of rational norm-curves,* American Journal of Mathematics, vol. 32 (1911), p. 167.

(*b*) *Basic quadrics meet* α *in quadrics apolar to* Q_α.

(*c*) *Basic* R^k*'s meet* α *in point sets* P_k^{k-1} *self polar as to* Q_α.

The quadric Q_α is defined as a locus of points by (*a*) and as an envelope of S_{k-2}'s by (*b*).

We shall need also the

LEMMA. *If the members of a pencil of* S_{k-1}*'s on an* S_{k-2} *and the* $k+2$ *base points are projective to the parameters of the base points on a basic* R^k, *the* S_{k-2} *cuts* R^k *in* $k-1$ *points.*

For if y is the parameter of the pencil, x the parameter on R^k and $y = x$ the projectivity, then the sets cut out on R^k by members of the pencil are determined by $(ay)(bx)^k = 0$. Since this must vanish $k+2$ times for $y = x$ we must have $(ax)(bx)^k \equiv 0$ whence $(ay)(bx)^k = (yx)\cdot(cx)^{k-1}$. Hence every S_{k-1} and therefore their common S_{k-2} cuts R^k in the $k-1$ points $(cx)^{k-1} = 0$.

According to (5) the pencil of S_{k-1}'s on the S_{k-2} containing $q_{k+4}, \cdots, q_{2k+2}$ and also in turn the points $q_1 = p_1, \cdots, q_{k+2} = p_{k+2}, q_{k+3}$ is projective to the set of parameters of $p_1, \cdots, p_{k+2}, p_{k+3}$ on the R^k through them. According to the lemma S_{k-2} must cut R^k in $k-1$ points and the particular $S_{k-1}(\alpha)$ which is on q_{k+3} must then contain p_{k+3}. But α is determined by $q_{k+3}, \cdots, q_{2k+2}$ and therefore contains all the points $p_{k+3}, \cdots, p_{2k+2}$. Moreover according to (*c*) the S_{k-2} in α which contains $q_{k+4}, \cdots, q_{2k+2}$ is the polar S_{k-2} of p_{k+3} as to Q_α. Hence

(10) *If the associated sets* P_{2k+2}^k *and* Q_{2k+2}^k *be placed so that the first* $k+2$ *points of each constitute the same base in* S_k *the remaining* k *points of each set lie in the same* S_{k-1}, α, *and form polar* k*-points of the quadric* Q_α *in* α *which is apolar to all the sections by* α *of basic quadrics.*

An inversion of the argument shows that this property implies the property of association. The construction in (10) can be used in connection with (6) to construct the associated sets P_n^k and Q_n^{k-k-2} whose norm is not zero. Suppose that the norm N of P_n^k is positive which according to (9) requires only a proper naming of the sets. Since $2k+2 = n+N$, P_n^k can be gotten by dropping say the last N points of P_{2k+2}^k in (10). If then Q_{2k+2}^k as constructed in (10) be projected from its last N points upon the S_{n-k-2} determined by the first $n-k-1$ base points we obtain a Q_n^{n-k-2} which is associated with P_n^k according to (6) and whose first $n-k-1$ points coincide with the corresponding points of P_n^k.

The striking feature of the P_{2k+2}^k with zero norm is that it may entirely coincide, i. e., be projective in the identical order, with its associated set. If this occurs P_{2k+2}^k will be called a *self-associated set.** Then in the identity

* The much discussed P_8^3, the base points of a net of quadrics, is usually called an "associated set."

(2) we can take (vq_i) to be $\lambda_i(up_i)$ and the identity becomes

$$(11)\qquad \lambda_1(up_1)^2 + \lambda_2(up_2)^2 + \cdots + \lambda_{2k+2}(up_{2k+2})^2 \equiv 0.$$

From this identity and from (10) we have

(12) *The self-associated set P^k_{2k+2} is defined by the property that all the quadrics on any $2k+1$ of the points pass through the remaining point. It is also defined by the property that the S_{k-1}, α, on any k of the points is cut by quadrics on the remdining $k+2$ points in sections all apolar to a definite quadric in α, Q_α, of which the k points form a self-polar k-edron. It depends upon $\frac{1}{2}k(k+1)$ absolute constants.**

After fixing the $k+2$ base points the number of absolute constants appears as the number k involved in the choice of α which also determines Q_α, and the number $\frac{1}{2}k(k-1)$ involved in the choice of a self polar k-edron of Q_α.

The P^1_4, any 4 points on a line, is a very special case which appears under (11). The P^2_6 on a conic is the next case. More generally any $2k+2$ points on the rational norm-curve R^k in S_k constitute a self-associated set [Cf. (*c*) above]. Again the $2k+2$ points cut out on an elliptic norm-curve, E^{k+1}, in S_k by a quadric are self-associated. This set is known to be general in S_2 and S_3 and can be proved to be the general set in S_4, but for higher dimensions it must be a special self-associated set since it contains only $2k+2$ absolute constants.

The association of point sets implies a mutual ordering of the sets. Self-association implies that the associated sets are also projective in the given order which will be referred to as the identical order. The associated sets may be projective in some other order and then the one set can be projected upon the other. Such a set will be said to be *self-associated in other than the identical order*. Let us consider the possible cases for the P^1_4 and the P^2_6. First we may note generally that

(13) *If a set P^k_{2k+2} on an R^k is self-associated in an order other than the identical one it is also self-projective in this order and conversely.*

For we have already noticed that such a set is self-associated in the identical order and two point sets each associated with a third are projective. Thus the identity (2) furnishes a convenient algebraic condition for the self-projectivity of such a set. From the known properties of the binary quartic we have

(14) *The general set P^1_4 is self-associated in four distinct orders; the harmonic P^1_4 in eight distinct orders; and the equianharmonic P^1_4 in twelve distinct orders.*

If the P^2_6 is on a conic, (13) leads to the self projective types of Bolza. Let us require then of P^2_6 that no two points coincide, that no three be on a line, and that the six be not on a conic. The identity (2) can be replaced by the

* For $k = 3$, cf. Serret, *Géométrie de direction*, pp. 313–16.

two identities

$$
(15) \qquad \begin{aligned} \lambda_1 (up_1)(uq_1) + \cdots + \lambda_6 (up_6)(uq_6) &\equiv 0, \\ \lambda_1 (p_1 q_1 x) + \cdots + \lambda_6 (p_6 q_6 x) &\equiv 0, \end{aligned}
$$

where the points q are some permutation of the points p. Hereafter the points p are indicated merely by their subscripts. The possible orders in which Q_6^2 can coincide with P_6^2 are $(12)(34)(56)$, $(12)(34)$, (12), $(123)(45)$, $(123)(456)$, (123), $(1234)(56)$, (1234), (12345), and (123456). Here, e. g., (123) indicates that $1, 2, 3, 4, 5, 6$ are associated with $2, 3, 1, 4, 5, 6$.

The order $(12)(34)(56)$ entails the identities:

$$
\begin{aligned} (\lambda_1 + \lambda_2)(u1)(u2) + (\lambda_3 + \lambda_4)(u3)(u4) + (\lambda_5 + \lambda_6)(u5)(u6) &\equiv 0, \\ (\lambda_1 - \lambda_2)(12x) + (\lambda_3 - \lambda_4)(34x) + (\lambda_5 - \lambda_6)(56x) &\equiv 0. \end{aligned}
$$

If all of the differences vanish the first identity requires that P_6^2 be the vertices of a four line. If only the first two differences vanish, 5 and 6 coincide. If only the first difference vanishes either two points coincide or $3, 4, 5, 6$ are on a line. All of these cases have been excluded. If none of the differences vanish, the lines $\overline{12}$, $\overline{34}$, $\overline{56}$, meet in a point and both identities can be satisfied by the choice, $\lambda_2 = -\lambda_1$, $\lambda_4 = -\lambda_3$, $\lambda_6 = -\lambda_5$.

The order $(12)(34)$ entails the identities:

$$
\begin{aligned} (\lambda_1 + \lambda_2)(u1)(u2) + (\lambda_3 + \lambda_4)(u3)(u4) + \lambda_5 (u5)^2 + \lambda_6 (u6)^2 &\equiv 0, \\ (\lambda_1 - \lambda_2)(12x) + (\lambda_3 - \lambda_4)(34x) &\equiv 0. \end{aligned}
$$

Unless $1, 2, 3, 4$ are on a line or unless two points coincide we must take $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$. According to the first identity the pairs $1, 2$ and $3, 4$ are apolar to a net of conics on 5 and 6. They must be either pairs on the conic C through $5, 6$ which with the line 56 makes up the jacobian of the net, or pairs on the line 56, or a pair on C and a pair on 56, all of which cases are excluded.

The order (12) if $1, 2$ do not coincide implies the identity:

$$
2\lambda_1 (u1)(u2) + \lambda_3 (u3)^2 + \cdots + \lambda_6 (u6)^2 \equiv 0.
$$

Thus $1, 2$ are apolar to the pencil of conics on $3, 4, 5, 6$; i. e., they correspond in the quadratic Cremona involution with the remaining four fixed points.

For the order (123) the second identity reads as follows:

$$
\lambda_1 (12x) + \lambda_2 (23x) + \lambda_3 (31x) \equiv 0,
$$

which requires that $1, 2, 3$ be on a line.

The first identity for the order $(123)(456)$ shows that the two triangles are self polar as to a conic which is necessarily proper unless two points coincide or three are on a line. Hence the six points are on a conic.

In the case of the order (123)(45) we can use the fact [cf. $C1$, p. 398] that a Cremona transformation T of the fifth order whose double singular points are P_6^2 can be constructed so that the double singular points of T^{-1} are Q_6^2. Then T^2 is a collineation which effects upon the points the square of the permutation (123)(45). If 1, 2, 3 are not on a line this collineation of period three has 4, 5, 6 for fixed triangle and P_6^2 is a pair of flex triangles of a cubic. We shall see below that this condition is sufficient.

A similar argument for the order (1234)(56) shows that a harmonic perspectivity with center $C \equiv \overline{\overline{13}\,\overline{24}}$ and axis $\overline{56}$ must interchange 1, 3 and also 2, 4. According to the first identity either all the conics apolar to the pairs 12, 23, 34, 41 must be apolar to the pair 56 or, if $\lambda_5 + \lambda_6 = 0$, all the conics apolar to 12, 23, 34 must be apolar to 41. The first case is impossible since then the lines $\overline{13}$ and $\overline{42}$ each have two poles as to such conics and the conics must be the line pairs on C harmonic with $\overline{13}$ and $\overline{42}$ which are not all apolar to 56. The second case also is impossible since there is in general only a pencil of conics apolar to the four pairs.

The case (12345) also is not possible if the P_6^2 is not on a conic. For T^2 is a cyclic collineation which is not an homology, say $x_1' = x_1$, $x_2' = \epsilon x_2$, $x_3' = \epsilon^4 x_3$ ($\epsilon = e^{2\pi i/5}$). Then the points $1, \cdots, 5$ are $1, \epsilon^\nu, \epsilon^{4\nu}$ ($\nu = 0, \cdots, 4$) while 6 must be 1, 0, 0. The parameters of $1, \cdots, 5$ on their conic are $t = 1, \epsilon^4, \epsilon^3, \epsilon^2, \epsilon$. The parameters of the pencil from 6 to $1, \cdots, 5$ are $t = 1, \epsilon^2, \epsilon^4, \epsilon, \epsilon^3$. These two sets in the proper order are not projective.

Taking up the case (1234) we find from T^2 that the pairs 1, 3 and 2, 4 correspond in a harmonic perspectivity with axis $\overline{56}$. If we take the points to be

$$1, 1, 1; \quad y_1, y_2, y_3; \quad -1, 1, 1; \quad -y_1, y_2, y_3; \quad 0, 1, 0; \quad 0, 0, 1,$$

we find from (2) that necessarily $y_2 + y_3 = 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$, $\lambda_5 = -4, \lambda_6 = 4$. Thus 5, 6 must harmonically separate the lines $\overline{13}$ and $\overline{42}$ which implies that P_6^2 is self-associated in the orders (1536) and (4526) as well. If it be required in addition that P_6^2 be self-associated in the order (2463) we find that $y_1 : y_2 : y_3 = -2 \pm \sqrt{5} : 1 : -1$. Then P_6^2 is the Clebsch six-point. If we take it in the canonical form

$$(up_\infty) = u_1, \qquad (up_\iota) = u_1 + \epsilon^{4i} u_2 + \epsilon^i u_3 \quad (i = 0, \cdots, 4),$$
$$\epsilon = e^{2\pi i/5},$$

the identity (2) for the permutation (1342) reads:

$$-5(u\infty)(v\infty) + (u0)(v0) + (u1)(v3) + (u2)(v1) + (u3)(v4) + (u4)(v2) \equiv 0.$$

Since this P_6^2 is self-projective in 60 ways we find by using the projectivities two more distinct ways of self-association, namely $(\infty 0)(34)(12)$ and $(\infty 23140)$.

Let us consider finally the order (123456). Now T^2 is a collineation of period three and T^3 is a Cremona transformation for which the points are associated in the order $(14)(25)(36)$ which requires that the lines $\overline{14}$, $\overline{25}$, $\overline{36}$ meet in a point. Thus the P_6^2 can be thrown into the form

$$1, 1, 1; \quad 1, \omega, \omega^2; \quad 1, \omega^2, \omega; \quad \rho, 1, 1; \quad \rho, \omega, \omega^2; \quad \rho, \omega^2, \omega;$$
$$\omega = e^{2\pi i/3}.$$

One verifies easily that the identity (2) holds for the values

$$\lambda_1 = \lambda_3 = \lambda_5 = -\lambda_2 = -\lambda_4 = -\lambda_6.$$

If $\rho = \omega$, we have two flex triangles. These are self-projective in 36 ways. By combining the projectivities with the present case of self-association we find that this P_6^2 is self-associated in the further types of order, (26), $(135)(26)$, and $(14)(36)(25)$. Hence

(16) *A* P_6^2 *for which no two points coincide, no three are on a line, and the six are not on a conic, can be associated with itself only in the following odd orders; (a) in* $(12)(34)(56)$ *if the lines* $\overline{12}$, $\overline{34}$, $\overline{56}$ *meet in a point* (3 *absolute constants); (b) in* (56) *if* 5 *and* 6 *correspond in the quadratic involution with fixed points at* 1, 2, 3, 4 (2 *absolute constants); (c) in* (1234) *if* $\overline{1, 3}$ *and* $\overline{4, 2}$ *are corresponding pairs in the reflexion with axis* 56 *and center* $\overline{\overline{13}}, \overline{\overline{42}}$ *and if the pair* 5, 6 *is apolar to the conic* $\overline{13}$, $\overline{42}$ (1 *absolute constant); (d) in* (123456) *if* 1, 3, 5 *and* 2, 4, 6 *are cyclically permuted by a collineation of period three with a fixed point on the lines* $\overline{14}$, $\overline{25}$, $\overline{36}$ (1 *absolute constant). Furthermore two flex triangles* 1, 3, 5 *and* 2, 4, 6 *are self-associated in the orders,* (123456), (26), $(135)(26)$, *and* $(14)(36)(25)$*; and a Clebsch six-point,* ∞, 0, 1, 2, 3, 4, *is self-associated in the orders,* (1342), $(\infty 0)(34)(12)$, *and* $(\infty 23140)$

If however P_6^2 *is on a conic it is self-associated in the identical order and in any other order according to which the six points or their parameters are self-projective. These types are furnished by Bolza's types of self-projective binary sextics.**

2. Irrational conditions for associated and self-associated point sets

Let P_n^k and Q_n^{n-k-2} be two associated sets for which, to fix ideas, the norm N of P_n^k is not greater than zero. If from the S_{k-j}, $u = c_0, c_1, \cdots, c_{k-j}$, determined by any $k - j + 1$ points of P_n^k numbered $c_0, \cdots, c_{k-j}$ we project

* A type of self-conjugate association is treated by E. Study, *Kürzeste Wege im komplexen Gebiet*, Mathematische Annalen, vol. 60 (1905), p. 321. For references to earlier articles on association see his footnote p. 348.

$2j$ points of P_n^k, say those numbered $a_1, \cdots, a_j, b_1, \cdots, b_j$, upon an S_{j-1} we obtain $2j$ points which are connected by the $\binom{2j}{j-1}$ determinant identities which pertain to S_{j-1}. If we denote by $P(i, \cdots, l)$ the $(k+1)$-rowed determinant formed from the coördinates of $(up_i), \cdots, (up_l)$, the corresponding identities in S_k have the form

$$\begin{aligned}(17)\quad & P(a_1, \cdots, a_j, u) P(b_1, \cdots, b_j, u) \\ & - P(b_1, a_1, \cdots, a_{j-1}, u) P(a_j, b_2, \cdots, b_j, u) + \cdots \\ & + (-1)^j P(a_2, \cdots, a_j, b_1, u) P(a_1, b_2, \cdots, b_j, u) = 0,\end{aligned}$$

the terms being obtained from the first by the cyclic advance of $a_1, \cdots, a_j, b_1$. If the remaining points of P_n^k are numbered $d_0, \cdots, d_{n-k-2-j}$ and if

$$v = \overline{d_0, \cdots, d_{n-k-2-j}}$$

is the $S_{n-k-2-j}$ in S_{n-k-2} which contains the corresponding points of Q_n^{n-k-2}, then according to the statement (7) we can replace the determinants above such as $P(a_1, \cdots, a_j, u)$ by determinants such as $Q(b_1, \cdots, b_j, v)$ if account be taken of the factor of proportionality (which may be neglected in any homogeneous relation) and the sign. If then each product in (17) be multiplied by the complementary product from Q_n^{n-k-2} and if the square root be taken, the identity is unaltered for proper choice of the signs of the radical and radicand. Then (17) takes the form

$$(18)\qquad \begin{aligned}&\sum \sqrt{P(a_1, \cdots, a_j, u) Q(b_1, \cdots, b_j, v)}\,. \\ &\sqrt{P(b_1, \cdots, b_j, u) Q(a_1, \cdots, a_j, v)} = 0,\end{aligned}$$

where $\sum$ refers to the cyclic advance of $a_1, \cdots, a_j, b_1$. We observe that if any change of sign occurs in the substitution of the complementary determinants as we pass from one term to the next this change must occur twice in the same product.

(19) *The coördinates of the associated sets P_n^k and Q_n^{n-k-2} satisfy the*

$$\binom{n}{k-j+1}\binom{n-(k-j+1)}{n-k-1-j}\binom{2j}{j-1}$$

irrational relations of type (18) *for $j = 2, \cdots, k+1$ if $N_{n,k} < 0$. A particular relation is determined by the choice of the $k-j+1$ points of P_n^k on u, of the $n-k-1-j$ complementary points of Q_n^{n-k-2} on v, and of the $j-1$ points $b_2, \cdots, b_j$ from the remaining $2j$ points of the sets. For any value of j these relations are sufficient to establish the association.*

Let us prove that the relations for $j = 2$ are sufficient. They require that if $p_1, \cdots, p_4$ be projected from $k-1$ points of P_n on u into points $\alpha_1, \cdots, \alpha_4$ of a line, and if $q_1, \cdots, q_4$ be projected from the complementary

$n-k-3$ points of Q_n^{n-k-2} on v into points $\beta_1, \cdots, \beta_4$ of a line, then

$$\sqrt{(\alpha_2\alpha_3)(\alpha_1\alpha_4)\cdot(\beta_2\beta_3)(\beta_1\beta_4)}+\sqrt{(\alpha_3\alpha_1)(\alpha_2\alpha_4)\cdot(\beta_3\beta_1)(\beta_2\beta_4)}$$
$$+\sqrt{(\alpha_1\alpha_2)(\alpha_3\alpha_4)\cdot(\beta_1\beta_2)(\beta_3\beta_4)}=0.$$

This can be written $\sqrt{x_1y_1}+\sqrt{x_2y_2}+\sqrt{x_3y_3}=0$ where $x_1+x_2+x_3=0$ and $y_1+y_2+y_3=0$. Rationalizing and eliminating x_1 and y_1 we find that $x_2/x_3=y_2/y_3$, i. e.,

$$\frac{(\alpha_3\alpha_1)(\alpha_2\alpha_4)}{(\alpha_1\alpha_2)(\alpha_3\alpha_4)}=\frac{(\beta_3\beta_1)(\beta_2\beta_4)}{(\beta_1\beta_2)(\beta_3\beta_4)}.$$

Hence the points α and the points β are projective. This being true of any four points and any choice of the points of projection we verify at once that the characteristic property (5) of associated sets holds for the given case. The value $j=k+1$ for $N_{n,k}=0$ is exceptional. In that case the relations (18) are satisfied when the associated sets P_{k2+2}^k and Q_{2k+2}^k are projective or when they are associated.

If for $N=0$ the set P_{2k+2}^k is a self-associated set then it can be brought into coincidence with its associated set Q_{2k+2}^k whence

(20) *The coördinates of the self-associated set* P_{2k+2}^k *satisfy the*

$$\tfrac{1}{2}\binom{2k+2}{k-j+1}\binom{k+j+1}{k-j+1}\binom{2j}{j-1}\qquad(j=2,\cdots,k)$$

irrational relations obtained from (18) *for* $Q=P$. *These conditions for a given value of j are sufficient for self-association.*

So far as the writer is aware these irrational conditions for association and self-association are novel. However for the P_6^2 on a conic one is sufficient and a condition closely related to that given above appears in connection with Kummer's surface.*

3. Invariants of point sets. The complete system of P_6^1

The definition of the rational integral invariants of the set P_n^1, i. e., of the binary form of order n, in terms of the particular points p can be generalized as follows:

(21) *A rational integral invariant of the point set* P_n^k *is a rational integral function of the* $\binom{n}{k+1}$ *determinants* $P(i_1, i_2, \cdots, i_{k+1})$ *which is* (*A*) *homogeneous and of the same degree in the coördinates of each of the n points and* (*B*) *unaltered in value however the points be permuted.*

Thus the general form of an invariant of the set is

$$(22)\qquad I=\sum\pi P(i_1, i_2, \cdots, i_{k+1})^{a_{i_1i_2\cdots i_{k+1}}},$$

* Cf. Hudson, *Kummer's Quartic Surface*, pp. 135–6 and elsewhere.

where according to (A) in each product π of $\sum$ the $\binom{n}{k+1}$ positive integers $a_{i_1 i_2 \cdots i_{k+1}}$ satisfy the diophantine system of n equations

$$\sum_j a_{i_1 i_2 \cdots i_{k+1}} = \nu \qquad (j = 1, \cdots, n). \tag{23}$$

Here $\sum_j$ indicates the sum of all the exponents a which have a subscript equal to j and ν is the *degree of the invariant*, i. e., its degree in the coördinates of a particular point. According to (B), $\sum$ in (22) refers to any sum of such products π, which is invariant under permutations of the points. Obviously under linear transformation I is unaltered to within a power of the determinant of the transformation. The index of this power is the *weight*, w, of I. Then

$$w = \frac{n\nu}{k+1}. \tag{24}$$

An *irrational integral invariant* of P_n^k is a function which satisfies all the requirements of (21) except (B). An *incomplete integral invariant* (rational or irrational as the case may be) of P_n^k is one which satisfies the proper requirements for a set P_n^k $(n' < n)$ contained in P_n^k. *Fractional* and *absolute* invariants are defined as is customary. Under the above definitions a cross-ratio of four points of a P_n^1 $(n > 4)$ would be described as an incomplete irrational absolute invariant of P_n^1.

From every invariant of any type of P_n^k there is obtained by passing to the complementary determinants an invariant J of the same type of the associated set Q_n^{n-k-2} and conversely. Since the weights are the same the degrees are connected by the relation $\nu_J/\nu_I = n/(k+1) - 1$.

(25) *The corresponding invariants I and J of the associated sets P_n^k, Q_n^{n-k-2} are equal to within a factor which is a power (whose index is the weight) of an undetermined constant. To a syzygy among the invariants I there corresponds a syzygy among the invariants J; to a complete system of invariants I, a complete system of invariants J.*

As an example of the determination of a complete system of invariants of an n-point, and as a means of introducing some formulæ for use in the next paragraph, let us take the case of a P_6^1 with coördinates p_{i1}, p_{i2} $(i = 1, \cdots, 6)$. The general invariant has the form

$$I = \sum \pi \, (ik)^{a_{ik}}, \qquad (ik) = (p_{i1} p_{k2} - p_{k1} p_{i2}),$$

where π satisfies the diophantine system

$$j^\circ. \quad a_{j1} + a_{j2} + \cdots + a_{j6} = \nu, \quad a_{jj} = 0, \quad a_{ij} = a_{ji} \qquad (j = 1, \cdots, 6).$$

Let us first prove that

7° *The product π can be expressed rationally and integrally in terms of the fifteen products of the type $(ij)(kl)(mn)$.* We shall say that π is "reducible" according to these fifteen "moduli."

(a) If $a_{12} = \nu$, then $a_{1i} = 0$, $a_{2i} = 0$ $(i = 3, \cdots, 6)$. From $4° + 5° + 6° - 3°$ we find that $a_{45} + a_{56} + a_{64} = \nu$.

(a_1) If $a_{45} = \nu$, then $a_{4i} = 0$, $a_{5i} = 0$ and from 6°, $a_{36} = \nu$ whence π is reducible mod. $(12)(45)(36)$.

(a_2) If $0 < a_{45} < \nu$ and $0 < a_{56} < \nu$ then $a_{36} = a_{34} = 0$ else π is reducible mod. $(12)(45)(36)$ or mod. $(12)(56)(34)$. Then $a_{35} = \nu$ which contradicts $0 < a_{45} < \nu$, due to 5°.

Thus π is reducible if any $a_{ik} = \nu$.

(b) Let $0 < a_{12} < \nu$. From $3° + 4° + 5° + 6° - 1° - 2°$ we have

$$a_{34} + a_{35} + a_{36} + a_{45} + a_{56} + a_{64} = \nu + a_{12}.$$

Hence at least two a's on the left are greater than zero. Let $0 < a_{34} < \nu$ whence $a_{56} = 0$ else π is reducible mod. $(12)(34)(56)$. Let $0 < a_{35} < \nu$ whence $a_{46} = 0$ else π is reducible. From $1° + 2° + 3° - 4° - 5° - 6°$ we now get $a_{12} + a_{13} + a_{23} - a_{45} = 0$, whence $0 < a_{45} < \nu$ and further $a_{36} = 0$ if π is not reducible. Now 6° reads as follows: $a_{61} + a_{62} = \nu$ and therefore $0 < a_{61} < \nu$ and $0 < a_{62} < \nu$. Thus if π is not reducible,

$$a_{13} = a_{14} = a_{15} = a_{23} = a_{24} = a_{25} = 0.$$

Then the solution of the system is

$$a_{12} = a_{16} = a_{26} = a_{34} = a_{35} = a_{45} = \nu/2.$$

Now π contains the factor

$$(12)(16)(26)(34)(35)(45) = (16)(26)(35)(45)[(13)(24) - (14)(23)]$$

and is reducible. The coefficients of the moduli thus introduced in π must satisfy a similar diophantine system for a smaller value of ν and the process can be repeated until the expression required in 7° is obtained. Hence

(26) *Any invariant I of P_6^1 is a rational integral function of the fifteen products $(ij)(kl)(mn)$, which is symmetric in the points.*

If now we form from these products the six functions of Joubert,*

$$(27)\quad \begin{aligned}
A &= (25)(13)(46) + (51)(42)(36) + (14)(35)(26) + (43)(21)(56) + (32)(54)(16),\\
B &= (53)(12)(46) + (14)(23)(56) + (25)(34)(16) + (31)(45)(26) + (42)(51)(36),\\
C &= (53)(41)(26) + (34)(25)(16) + (42)(13)(56) + (21)(54)(36) + (15)(32)(46),\\
D &= (45)(31)(26) + (53)(24)(16) + (41)(25)(36) + (32)(15)(46) + (21)(43)(56),\\
E &= (31)(24)(56) + (12)(53)(46) + (25)(41)(36) + (54)(32)(16) + (43)(15)(26),\\
F &= (42)(35)(16) + (23)(14)(56) + (31)(52)(46) + (15)(43)(26) + (54)(21)(36),
\end{aligned}$$

* These functions have been used by Joubert and Richmond (for references see $C2$) to obtain invariants of the sextic but not to prove the completeness of the system.

we verify at once that an even permutation of the points effects an even permutation of the functions and that an odd permutation of the points effects an odd permutation of the functions accompanied by a change of sign throughout. A set of parallel generating permutations is

$$(12),\ (23456) : (AD)(BE)(CF),\ (ADBFE). \tag{28}$$

The six points and functions define the two sets of six conjugate ikosahedral subgroups in the even permutation group, g_{360}. The fundamental binary identity leads at once to the relation $A + D = 4(43)(21)(56)$ and by applying (28) we get the conjugate set,

$$\begin{array}{ll} A + B = 4(51)(43)(26), & B + C = 4(34)(25)(16), \\ A + C = 4(53)(41)(26), & B + D = 4(31)(45)(26), \\ A + D = 4(43)(21)(56), & B + E = 4(12)(53)(46), \\ A + E = 4(32)(54)(16), & B + F = 4(23)(14)(56), \\ A + F = 4(25)(13)(46), & C + D = 4(15)(32)(46), \\ \multicolumn{2}{c}{C + E = 4(13)(42)(56),} \\ \multicolumn{2}{c}{C + F = 4(54)(21)(36),} \\ \multicolumn{2}{c}{D + E = 4(25)(41)(36),} \\ \multicolumn{2}{c}{D + F = 4(24)(53)(16),} \\ \multicolumn{2}{c}{E + F = 4(15)(43)(26).} \end{array} \tag{29}$$

Thus the fifteen products can be expressed rationally and integrally in terms of $A, \cdots, F$. Then the half symmetric invariants are rational and integral in $\sum A, \sum A^2, \sum A^3, \sum A^4, \sum A^5, \sum A^6$ and $\pi(A - B)$. Of these $\sum A, \sum A^3, \sum A^5$ change sign under odd permutation of the points and therefore must contain $\sqrt{D} = \pi(ik)$ as a factor. The weight of the first two is too low to permit this whence $\sum A = \sum A^3 = 0$ while $\sum A^5 = \lambda \sqrt{D}$. This shows that

(30) *The invariants* $\sum A^2, \sum A^4, \sum A^6, (\sum A^5)^2$, *and* $\pi(A - B)$ *constitute a complete system of rational integral invariants of* P_6^1, *i. e. of the binary sextic. The square of the last is rationally expressible in terms of the others.*

By using (29) we find that the factor $A - B$ of the skew invariant $\pi(A - B)$ has the value

$$A - B = (A + C) - (B + C) = \cdots = 4[(53)(41)(26) - (34)(25)(16)] = \cdots.$$

If it vanishes the pairs 15, 24, 36 are in an involution.

By means of this method it is quite easy to derive and to prove the completeness of the system of invariants and covariants of the binary cubic and quartic.

4. Lines and tritangent planes of a cubic surface mapped from a plane by cubic curves on six points

The principle of transference of Clebsch applied to the functions of Joubert leads to the result:

(31) *If in the functions, $A, \cdots, F$, each binary determinant (ik) be replaced by the ternary determinant (ikx), there arise six cubic curves, $a = 0, \cdots, f = 0$, on six points, $p_1, \cdots, p_6$ of a plane π, which are connected by the identities* $\sum a = 0,\ \sum a^3 = 0,\ \sum a^5 = \lambda\pi(ikx),\ a + b = 4(51x)(42x)(36x),\ \cdots$.

Six cubic curves on six points are connected by two linear relations. One of these is $\sum a = 0$. If we take the other to be $\sum \bar{a}a = 0$, the coefficients $\bar{a}, \cdots, \bar{f}$ can be fixed to within a factor of proportionality by the requirement that $\sum \bar{a} = 0$.

Let us denote by (ij, kl, mn) the ternary determinant whose vanishing expresses that the lines (ijx), (klx), (mnx) meet in a point. Then the fifteen relations

(32)
$$\begin{array}{ll}
\bar{a} - \bar{b} = (15, 24, 36), & \bar{b} - \bar{c} = (25, 34, 16), \\
\bar{a} - \bar{c} = (14, 35, 26), & \bar{b} - \bar{d} = (13, 54, 26), \\
\bar{a} - \bar{d} = (12, 43, 56), & \bar{b} - \bar{e} = (12, 35, 46), \\
\bar{a} - \bar{e} = (23, 45, 16), & \bar{b} - \bar{f} = (14, 23, 56), \\
\bar{a} - \bar{f} = (13, 52, 46), & \bar{c} - \bar{d} = (15, 32, 46),
\end{array}$$
$$\begin{aligned}
\bar{c} - \bar{e} &= (13, 24, 56), \\
\bar{c} - \bar{f} &= (12, 45, 36), \\
\bar{d} - \bar{e} &= (14, 52, 36), \\
\bar{d} - \bar{f} &= (24, 35, 16), \\
\bar{e} - \bar{f} &= (15, 34, 26),
\end{aligned}$$

of which the first is assumed and the others deduced from it by applying the parallel generating permutations (28) can be shown to be consistent. For from the identities

$$(153)(246) - (156)(234) + (136)(245) - (536)(241) = 0,$$
$$(136)(425) - (134)(625) + (164)(325) - (364)(125) = 0,$$

we get by addition the identity

$$(153)(246) - (156)(243) + (251)(346) - (256)(341) + (142)(536) - (146)(532) = 0,$$

whose terms pair off and give rise to

$$(33)\qquad \left\{\begin{matrix}123\\465\end{matrix}\right\} = (14, 26, 35) + (16, 25, 34) + (15, 24, 36) = 0.$$

This identity written according to (32) has the form

$$(\bar{a} - \bar{b}) + (\bar{b} - \bar{c}) + (\bar{c} - \bar{a}) = 0$$

and the consistency of (32) is verified throughout by the use of identities similar to (33). The six functions $\bar{a}, \cdots, \bar{f}$ can be obtained from the first column of (32) and the equation $\sum\bar{a} = 0$.

If we subtract the two identities above we find that

$$\begin{aligned} 2(136)(245) &= -(135)(264) + (134)(265) + (165)(234) \\ &\quad - (164)(235) + (365)(124) - (364)(125) \\ &= (13, 26, 45) + (16, 23, 54) + (36, 12, 54) \\ &= (\bar{b} - \bar{d}) + (\bar{e} - \bar{a}) + (\bar{f} - \bar{c}) \\ &= -2(\bar{a} + \bar{c} + \bar{d}) = 2(\bar{b} + \bar{e} + \bar{f}). \end{aligned}$$

By permutation we obtain the ten relations

$$(34)\qquad \begin{aligned} \bar{d} + \bar{e} + \bar{f} &= -(123)(456), & \bar{b} + \bar{d} + \bar{f} &= -(125)(436), \\ \bar{c} + \bar{e} + \bar{f} &= -(146)(253), & \bar{b} + \bar{d} + \bar{e} &= -(156)(234), \\ \bar{c} + \bar{d} + \bar{f} &= -(134)(265), & \bar{b} + \bar{c} + \bar{f} &= -(153)(246), \\ \bar{c} + \bar{d} + \bar{e} &= -(126)(345), & \bar{b} + \bar{c} + \bar{e} &= -(145)(326), \\ \bar{b} + \bar{e} + \bar{f} &= -(136)(254), & \bar{b} + \bar{c} + \bar{d} &= -(142)(356). \end{aligned}$$

Ten similar relations are obtained by using $\sum\bar{a} = 0$.

Solving the equations (32) by using $\sum\bar{a} = 0$ we find that

$$(35)\qquad \begin{aligned} 6\bar{a} &= (15, 24, 36)+(14, 35, 26)+(12, 43, 56)+(23, 45, 16)+(13, 52, 46), \\ 6\bar{b} &= -(15, 24, 36) +(25, 34, 16)+(13, 54, 26)+(12, 35, 46)+(14, 23, 56), \\ 6\bar{c} &= -(14, 35, 26)-(25, 34, 16) +(15, 32, 46)+(13, 24, 56)+(12, 45, 36), \\ 6\bar{d} &= -(12, 43, 56)-(13, 54, 26)-(15, 32, 46) +(14, 52, 36)+(24, 35, 16), \\ 6\bar{e} &= -(23, 45, 16)-(12, 35, 46)-(13, 24, 56)-(14, 52, 36) +(15, 34, 26), \\ 6\bar{f} &= -(13, 52, 46)-(14, 23, 56)-(12, 45, 36)-(24, 35, 16)-(15, 34, 26). \end{aligned}$$

It is to be noted that odd permutations of the points lead to odd permutations of $\bar{a}, \cdots, \bar{f}$ *without* change of sign and that $\bar{a}, \cdots, \bar{f}$ arise from $A, \cdots, F$ in (27) by replacing the product $(ij)(kl)(mn)$ in the order there written by (ij, kl, mn).

Consider the cubic $C = \bar{a}a + \cdots + \bar{f}f = 0$. It at most changes sign when the points are permuted; it is on the six points; and if it is on the meet of the lines $\overline{12}$ and $\overline{34}$ it must be on the 45 points of that kind, must contain the 15 lines (ikx), and therefore must vanish identically. From (31) and (29) we find that if $(12x) = (34x) = 0$, then

$$a + d = b + c = b + e = c + f = e + f = 0.$$

Then C takes the form

$$(\bar{a} - \bar{d})a + (\bar{b} + \bar{f} - \bar{c} - \bar{e})b = (\bar{a} - \bar{d})(a + b) + (\bar{b} + \bar{f} + \bar{d} - \bar{a} - \bar{c} - \bar{e})b.$$

If we substitute here from (34) and (32), replace x by $\overline{\overline{1234}}$ by using (29) and (27), expand all terms like (ij, kl, mn) except $(12, 34, 56)$, and factor out $(123)(124)(134)(234)(125)(346)$, then C takes the form,

$$4(12, 34, 56) + 2[(56, 12, 34) - (126)(345) + (512)(634)] = 4(12, 43, 56) + 4(12, 34, 56) = 0.$$

Thus C vanishes identically and the required second identity connecting $a, \cdots, f$ is found.

(36) *If in the functions* $A, \cdots, F$ *as written in* (27) *each product*

$$(ij)(kl)(mn)$$

be replaced by the ternary determinant (ij, kl, mn), *then the functions take values* $6\bar{a}, \cdots, 6\bar{f}$ *such that* $\sum \bar{a} = 0$ *and* $\sum \bar{a}a = 0$.

The fact that cubic curves on the six points p of the plane π map the plane on a general cubic surface $C^{(3)}$ is well known but it seems that an explicit equation of $C^{(3)}$ in terms of the coördinates of the points p has not been given hitherto. Let us then derive the equations not only of $C^{(3)}$ but also of its 45 tritangent planes and therefore of its 27 lines. The six sets of directions about the points p map into skew lines $a_1, \cdots, a_6$ of $C^{(3)}$. The six conics on five of the six points map into six skew lines $b_1, \cdots, b_6$ of $C^{(3)}$. The lines a_i and b_i form a "double six," i. e., a_i meets b_j if $i \neq j$. The 15 lines (ijx) of π map into the remaining 15 lines c_{ij} of $C^{(3)}$. The 45 tritangent planes of $C^{(3)}$ separate into two sets with reference to the double six. Fifteen planes cut $C^{(3)}$ in the maps of cubic curves on π of the type $(ijx)(klx)(mnx)$; thirty in the maps of cubic curves made up of a line (ijx) and a conic on i, k, l, m, n.

As regards the equation of $C^{(3)}$ we have at once:

(37) *The equation of* $C^{(3)}$ *in Cremona's hexahedral form is* $\sum a^3 = 0$ *where the variables are subject to the relations,* $\sum a = 0$, *and* $\sum \bar{a}a = 0$.

From (31) there follows at once that 15 tritangent planes of $C^{(3)}$ are of the type $a + d = 0$, and that 15 lines of $C^{(3)}$ are of the type

$$a + d = b + e = c + f = 0.$$

The map of this particular line on π is $(12x) = 0$. On this line of $C^{(3)}$ there are five tritangent planes, three of which have just been given. The maps of the two others on π are the conics on 1, 3, 4, 5, 6 and on 2, 3, 4, 5, 6 each taken with the line $(12x) = 0$. One finds easily that the equations of the two maps are

$$(38)\qquad \begin{aligned} &(531)(461)(a + d) - (341)(561)(b + e) = 0,\\ &(532)(462)(a + d) - (342)(562)(b + e) = 0. \end{aligned}$$

Since only products of the types $(ijk)(lmn)$ can be expressed in terms of $\bar{a}, \cdots, \bar{f}$ by means of (34) we must multiply the two cubics (38) together. By an easy manipulation of the term in $(a + d)(b + e)$ we get:

$$(39)\qquad \begin{aligned} &(a + d)^2[(531)(462) \cdot (532)(461)]\\ &\quad + (b + e)^2[(341)(562) \cdot (342)(561)]\\ &\quad - (a + d)(b + e)[(531)(462) \cdot (341)(562)\\ &\quad + (532)(461) \cdot (342)(561) + (563)(142) \cdot (512)(436)] = 0. \end{aligned}$$

Here the coefficient of $(a + d)(b + e)$ is worth noting. If we set

$$(40)\qquad a_2 = \sum \bar{a}\bar{b}, \qquad a_3 = \sum \bar{a}\bar{b}\bar{c}, \qquad \cdots, \qquad a_6 = \bar{a}\bar{b}\bar{c}\bar{d}\bar{e}\bar{f},$$

and then express this coefficient in terms of $\bar{a}, \cdots, \bar{f}$, it is

$$(\bar{b} + \bar{c} + \bar{f})(\bar{d} + \bar{c} + \bar{f}) - (\bar{c} + \bar{e} + \bar{f})(\bar{b} + \bar{d} + \bar{e}) \\ - (\bar{b} + \bar{c} + \bar{d})(\bar{b} + \bar{d} + \bar{f}),$$

which takes the following form symmetric in $\bar{c}$ and $\bar{f}$:

$$(\bar{b} + \bar{c} + \bar{f})(\bar{d} + \bar{c} + \bar{f}) + (\bar{e} + \bar{c} + \bar{f})(\bar{a} + \bar{c} + \bar{f}) \\ + \tfrac{1}{2}\{(\bar{b} + \bar{c} + \bar{d})(\bar{a} + \bar{c} + \bar{e}) + (\bar{b} + \bar{d} + \bar{f})(\bar{a} + \bar{e} + \bar{f})\}.$$

This by multiplication and use of $\sum \bar{a} = 0$ becomes

$$(41)\qquad \overline{cf} = a_2 + 2(\bar{c}^2 + \bar{f}^2 + \bar{c}\bar{f}).$$

Again by virtue of $\sum a = 0$ the quadratic (39) can be written in terms of $(a + d)^2$, $(b + e)^2$, $(c + f)^2$ and the coefficient of $(c + f)^2$ is half that of

$(a+d)(b+e)$ in (39). Hence (39) becomes

$$(42)\qquad \overline{ad}(a+d)^2+\overline{be}(b+e)^2+\overline{cf}(c+f)^2=0.$$

The solution of (42) can be written in three ways,

$$(43)\qquad \begin{aligned}(\overline{be}-d_2)(b+e)-(\overline{cf}+d_2)(c+f)&=0,\\ (\overline{cf}-d_2)(c+f)-(\overline{ad}+d_2)(a+d)&=0,\\ (\overline{ad}-d_2)(a+d)-(\overline{be}+d_2)(b+e)&=0,\end{aligned}$$

which are equivalent due to the fact that

$$(44)\qquad -d_2^2=\overline{be}\,\overline{cf}+\overline{cf}\,\overline{ad}+\overline{ad}\,\overline{be}.$$

The two planes coincide if the irrationality d_2 vanishes. Then the conics on $1, 3, 4, 5, 6$ and $2, 3, 4, 5, 6$ must coincide and the six points must be on a conic. This condition, or rather its square, must be symmetric in the six points whence $d_2^2=\lambda a_2^2+\mu a_4$. If $\overline{be}\,\overline{cf}$ be multiplied out and symmetrized by using 45 terms we find that

$$(45)\qquad d_2^2=a_2^2-4a_4.$$

In order to definitely fix the sign of d_2 and thereby isolate the members of the pair of planes in (42) let us set for brevity

$$\lambda=(531)(461),\qquad \mu=(532)(462),\qquad \nu=(341)(561),$$
$$\rho=(342)(562).$$

Then comparing (38) with (42) we find that

$$(46)\qquad \tfrac{1}{2}(\overline{ad}+\overline{cf})=\lambda\mu,\qquad \tfrac{1}{2}(\overline{be}+\overline{cf})=\nu\rho,\qquad -\overline{cf}=\lambda\rho+\mu\nu.$$

Now $\mu\nu-\lambda\rho=0$ is clearly the condition that the six points be on a conic so that $\mu\nu-\lambda\rho=kd_2$, where d_2 is defined in (44). Squaring and using the values (46) we find that $k^2d_2^2=-\overline{be}\,\overline{cf}-\overline{cf}\,\overline{ad}-\overline{ad}\,\overline{be}$. Comparing this with (44) let us take $k=1$. Again $\overline{ad}=2\lambda\mu+\lambda\rho+\mu\nu$, $\overline{be}=2\nu\rho+\lambda\rho+\mu\nu$, $\overline{cf}=-\lambda\rho-\mu\nu$, $d_2=\mu\nu-\lambda\rho$, and (43) reduces to $\lambda(a+d)-\nu(b+e)$ which is the first of the planes in (38).

(47) *The* 45 *tritangent planes of* $C^{(3)}$ *comprise* 15 *of the type*

$$a+d=0$$

which maps into $(12x)(34x)(56x)$, *and* 30 *of the type given by* (43) *which maps into* $(12x)(13456x^2)$, *where the symbols* a_2, $\overline{cf}$, *and* d_2^2 *are defined by* (40), (41), (44), *and* (45), *while the sign of* d_2 *is defined by*

$$d_2=\begin{vmatrix}(341)(561) & (531)(461)\\ (342)(562) & (532)(462)\end{vmatrix}.$$

All the tritangent planes are deduced from these typical ones by the corresponding permutations (28) *of the letters and figures, an odd permutation being accompanied by a change of sign of* d_2.

5. The complete system of P_6^2

The P_6^2 is the simplest P_n^k such that neither P_n^k nor its associated Q_n^{n-k-2} can be given by a binary form. Thus there is a certain novelty in its complete system though a sketch of the results of these allied §§ 4, 5 has appeared in an earlier paper.*

By definition the general invariant I of P_6^2 is

$$I = \sum \pi \, (p_i \, p_k \, p_j)^{a_{ijk}}$$

where $\sum$ is symmetric in the six points and π contains each point to the degree ν.

The diophantine system contains six equations, $1°, \cdots, 6°$, the first of which reads as follows:

$1°\quad a_{123} + a_{124} + a_{125} + a_{126} + a_{134} + a_{135} + a_{136} + a_{145} + a_{146} + a_{156} = \nu.$

When $\nu = 1$ we have ten solutions of the type

$$(a)\qquad (123)(456), \qquad (124)(356), \qquad \cdots.$$

When $\nu = 2$ we have thirty solutions of the type

$$(b)\qquad (123)(145)(256)(364).$$

Six of these contain (123), namely the six which occur in the terms of the determinant

$$(123)\begin{vmatrix} (156) & (164) & (145) \\ (256) & (264) & (245) \\ (356) & (364) & (345) \end{vmatrix}.$$

We shall now prove that every product π subject to the diophantine system can be expressed rationally and integrally in terms of the moduli (a) and (b). To prove this we have to show only that π contains a factor which can be so expressed. The remaining factor must then satisfy the system for a smaller value of ν and the proof is completed by exhaustion.

Let $a_{123} > 0$; if then $a_{456} > 0$, π contains a factor (a).

Let $a_{123} > 0$ and $a_{456} = 0$. By adding the first three and the last three of equations $1°, \cdots, 6°$ and by adding and subtracting the resulting equations we find that

$$\begin{aligned} a_{123} + \sum a_{rs,\,t} + \sum a_{mn,\sigma} &= 2\nu \\ 3a_{123} + \sum a_{rs,\,t} - \sum a_{mn,\sigma} &= 0 \end{aligned} \qquad \begin{pmatrix} rs = 23, 31, 12; & t = 4, 5, 6 \\ mn = 56, 64, 45; & \sigma = 1, 2, 3 \end{pmatrix}.$$

Hence $\sum a_{rs,\,t} = \nu - 2a_{123}$, $\sum a_{mn,\sigma} = \nu + a_{123}$.

*Johns Hopkins University Circular, no. 232 (1911), p. 59.

If none of the values $a_{mn,\sigma}$ which have the subscript $\sigma = 1$ are effective (i. e., greater than zero) then $\sum a_{1ik} < a_{123} + \sum a_{rs,\,t} = n - a_{123} < \nu$ which contradicts 1°. Hence of the effective values $a_{mn,\sigma}$ there is at least one for which $\sigma = 1$, and similarly at least one for which $\sigma = 2$, and at least one for which $\sigma = 3$.

If none of the values $a_{mn,\sigma}$ for which $mn = 56$ are effective then all the effective ones have $m = 4$ or $n = 4$. Hence $\sum a_{4ik} > \sum a_{mn,\sigma} > \nu$ which contradicts 4°. Thus of the effective values $a_{mn,\,\sigma}$ there is one at least for which $mn = 56$, one at least for which $mn = 64$, and one at least for which $mn = 45$.

If then there are only three effective values $a_{mn,\sigma}$ π must contain as a factor one term of the above determinant, i. e., a factor of type (b).

If there are four effective values $a_{mn,\sigma}$ and no term of the determinant is a factor then π must contain a factor of the type

$$(123)(156)(256)(364)(345).$$

Then unless $a_{642} = a_{542} = a_{415} = a_{416} = a_{423} = a_{413} = 0$, π contains a factor of type (a) or (b). Taking these to be zero, 4° becomes

$$\sum a_{4ik} = a_{412} + a_{435} + a_{436} = \nu.$$

If $a_{412} = 0$ then $a_{435} + a_{436} = \nu$ and $a_{123} + {}_{435} + a_{436} > \nu$ which contradicts 3°. Hence $a_{412} > 0$ and π contains the factor

$$(123)(124)(345)(346)(561)(562)$$
$$= (124)(346)(561)(562)[(431)(253) + (234)(153)]$$

which is evidently reducible.

A fortiori if there are more than four effective values $a_{mn,\sigma}$, π is reducible. Hence every invariant I is a rational integral function of the moduli (a) and (b).

We see at once that the two moduli of type (b),

$$\alpha = (123)(145)(256)(364), \qquad \beta = (125)(143)(236)(564),$$

are closely related. In fact, $d_2 = \alpha - \beta = 0$ is the condition that the six points be on a conic. Thus the 30 moduli (b) divide into 15 pairs such that d_2 can be expressed as the difference of the members of a pair. Moreover

$$\begin{aligned}\alpha + \beta &= (256)(364)[(134)(215) + (142)(315)] \\ &\quad + (236)(564)[(154)(213) + (142)(513)] \\ &= (134)(256)\cdot(215)(364) + (236)(154)\cdot(564)(213) \\ &\quad + (142)(315)[(256)(364) - (236)(564)] \\ &= (134)(256)\cdot(215)(364) + (236)(154)\cdot(564)(213) \\ &\quad + (142)(365)\cdot(315)(246).\end{aligned}$$

Hence the thirty moduli (b) or the fifteen pairs α, β can be expressed rationally and integrally in terms of d_2 and the ten moduli (a). Moreover from (34) and (35) the ten moduli (a) and d_2^2 can be similarly expressed in terms of $\bar{a}, \cdots, \bar{f}$ whence

$$I = r(\bar{a}, \cdots, \bar{f}) + d_2 s(\bar{a}, \cdots, \bar{f})$$

where r and s are rational integral functions of their arguments. Applying the 360 even permutations of the points and adding the equations we find that

$$I = r_1(a_2, \cdots, a_6) + \sqrt{d}\, r_2(a_2, \cdots, a_6) \\ + d_2 s_1(a_2, \cdots, a_6) + d_2\sqrt{d}\, s_2(a_2, \cdots, a_6),$$

where $a_2, \cdots, a_6$ are defined in (40) and

(48) $$d = \pi(\bar{a} - \bar{b})^2.$$

Applying an even permutation we find that

$$I = r_1 - \sqrt{d}\, r_2 - d_2 s_1 + d_2\sqrt{d}\, s_2,$$

whence

$$2I = r_1(a_2, \cdots, a_6) + d_2\sqrt{d}\, s_2(a_2, \cdots, a_6).$$

(49) *The invariants a_2, a_3, a_4, a_5, a_6, and $d_2\sqrt{d}$ constitute a rational and integral complete system of invariants of P_6^2. There is but one syzygy connecting them—that which expresses the square of $d_2\sqrt{d}$ in terms of the others.*

Since $d_2^2 = a_2^2 - 4a_4$ the irrational invariants which are half symmetric can all be rationally expressed in terms of a_2, d_2, a_3, a_5, a_6, and $\sqrt{d}$.

The occurrence of the invariant $d_2\sqrt{d}$ in (49)—the vanishing of whose irrational factors d_2 and $\sqrt{d}$ have quite different projective interpretations—indicates that the $g_{6!}$ of permutations of the points is not the proper group to use in connection with P_6^2. One may use the $g_{\frac{1}{2}6!}$ whose invariants have just been given, or the $g_{2\cdot 6!}$ mentioned in § 10 whose invariants are $a_2, \cdots, a_6$.

6. Mapping of P_n^k's upon points of $\Sigma_{k(n-k-2)}$

In this paragraph various methods for mapping the class of projectively equivalent sets P_n^k upon a point P of a space $\Sigma_{k(n-k-2)}$ are discussed. One of the simplest type is selected and utilized later to study non-projective operations on the set P_n^k (such as a change in the order of its points) as operations on the point P.

A given point set P_n^k is projectively defined only when the points are ordered since it is in general self-projective only in the identical order. If there be given in S_k a basis, $(ub_1) = 0, \cdots, (ub_{k+2}) = 0$, consisting either of the reference points and the unit point (the "canonical basis") or of any other properly chosen $k + 2$ points, the set P_n^k can be linearly transformed so that the first $k+2$ points, $(up_1), \cdots, (up_{k+2})$, fall in order upon $(ub_1), \cdots, (ub_{k+2})$. The remaining $n-k-2$ points then take positions, $(ux^{(1)})=0, \cdots, (ux^{(n-k-2)})=0$, which vary as P_n^k varies through projectively distinct classes.

In a space $\Sigma_{k(n-k-2)}$ choose a space $T^{(j)}_{k(n-k-3)-1}$ determined by the $k+1$ linear equations, $(\alpha_1^{(j)}y)=0, \cdots, (\alpha_{k+1}^{(j)}y)=0$. On $T^{(j)}$ there are ∞^k spaces $R^{(j)}_{k(n-k-3)}$ which can be put into one-to-one correspondence with the points of S_k. This is effected most simply by using the canonical basis in S_k and requiring that the particular $R^{(j)}$ defined by $(\alpha_i^{(j)}y)=0$ $(i=1, \cdots, l-1, l+1, \cdots, k+1)$ shall correspond to the reference point (ub_l) of S_k, and that the particular $R^{(j)}$ defined by $(\alpha_i^{(j)}y)=(\alpha_l^{(j)}y)$ $(i, l=1, \cdots, k+1)$ shall correspond to the unit point of S_k. Then the equation of the $R^{(j)}_{k(n-k-3)}$ which corresponds to the point $(ux^{(j)})=0$ in S_k is

$$(50)\qquad \left\| \begin{matrix} (\alpha_1^{(j)}y) & (\alpha_2^{(j)}y) & \cdots & (\alpha_{k+1}^{(j)}y) \\ x_1^{(j)} & x_2^{(j)} & \cdots & x_{k+1}^{(j)} \end{matrix} \right\| = 0 \qquad (j=1, \cdots, n-k-2).$$

Thus by choosing $n-k-2$ spaces $T^{(j)}$ in general position in $\Sigma_{k(n-k-2)}$ and by passing through each a space $R^{(j)}_{k(n-k-3)}$ determined by one of the last $n-k-2$ points of P_n^k we obtain a point P of Σ, the meet of the $n-k-2$ spaces $R^{(j)}$, which is the *map in* Σ *of the set* P_n^k *in* S_k. Conversely a general point P of Σ determines an $R^{(j)}$ on $T^{(j)}$ and therefore a point $(ux^{(j)})=0$ in S_k. These $n-k-2$ points together with the selected base is the P_n^k of S_k which corresponds to P in Σ. If the point P of Σ with coördinates y be given the coördinate $x_i^{(j)}$ of the point p_{k+2+j} of P^k is $(\alpha_i^{(j)}y)$ $(i=1, \cdots, k+1; j=1, \cdots, n-k-2)$. If the coördinates $x_i^{(j)}$ of P_n^k are given, the coördinates y of P are obtained by solving $k(n-k-2)$ independent linear equations obtained from the matrices (50).

We have supposed that $n \geqq k+3$. If $n=k+3$ the above method of mapping fails. We have then merely to transform the first $k+2$ points of P_{k+3}^k to a selected basis and to take the last point as the map in $\Sigma_k = S_k$ of P_{k+3}^k. For the associated set Q_{k+3}^1 however the above construction is still valid unless $k=1$.

The singular point sets P_n^k of this mapping are (1) those sets whose first $k+2$ points do not form a basis and (2) those sets for which the $n-k-2$ spaces $R^{(j)}$ meet in a line. These spaces meet in a line L if and only if the line L cuts each space $T^{(j)}$. That a line should meet a particular $T^{(j)}$ is k conditions whence there are $\infty^{k(n-k-2)-2}$ such lines which lie on a spread M in Σ. To determine the order of M let y be a point on such a line L and in Σ project from y upon a particular space say $T^{(1)}$ each of the other spaces $T^{(j)}$, i. e., pass an $R^{(j)}$ through y and $T^{(j)}$ cutting $T^{(1)}$ in an $S^{(1j)}_{k(n-k-3)-(k+1)}$. It is necessary and sufficient that in $T^{(1)}$ the $n-k-3$ spaces $S^{(1j)}$ be on a point. This single condition is of order $n-k-3$ in y since in $T^{(1)}$ the coördinates of $S^{(1j)}$ are linear in y. For if $S^{(1j)}$ be required to meet a given S_{k-1} in $T^{(1)}$ in a point, then the S_{k-1} and $R^{(j)}$ meeting in a point determine in Σ an $S_{k(n-k-2)-1}$, the locus of possible positions of y. Hence the locus of lines L is a spread M^{n-k-3} of order $n-k-3$.

On the other hand the singular points P in Σ of the mapping are the points of the spaces $T^{(j)}$.

If we use a similar scheme of coördinate spaces $T'^{(j)}$ in a space Σ' (which may or may not be superposed upon Σ) in order to map the sets P_n^k, then the point y of Σ and the point y' of Σ' which map the same P_n^k are corresponding points in a Cremona transformation C. This transformation corresponds to a change of coördinates in the mapping. It is a particular case of the transformation obtained from bilinear forms. In fact its equations are

$$(51)\qquad \left\| \begin{matrix} (\alpha_1^{(j)}y) & (\alpha_2^{(j)}y) & \cdots & (\alpha_{k+1}^{j}y) \\ (\alpha_1'^{(j)}y') & (\alpha_2'^{(j)}y') & \cdots & (\alpha_{k+1}'^{(j)}y') \end{matrix} \right\| = 0$$

$$(j = 1, \cdots, n-k-2).$$

The order of such a transformation is in general $k(n-k-2)$ but due to the matrix form of the bilinear relations a reduction occurs here and the order of C is $n-k-2$. For as y runs over a line L in Σ the spaces $R^{(j)}$ are brought into projective relation. Hence in Σ' we get the locus of the intersections of corresponding members of $n-k-2$ projective pencils of $R'^{(j)}$'s. Each pencil lies in an $S'_{k(n-k+3)+1}$ and these have a common S'_{n-k-2}. In this common space we have $n-k-2$ projective pencils of S'_{n-k-3}'s which generate a rational norm-curve of order $n-k-2$, the correspondent of L, which meets each space $T'^{(j)}$ in $n-k-3$ points.

The singular points of C in Σ are (1) the points of the spaces $T^{(j)}$ and (2) the points of Σ for which the spaces $R^{(j)}$ are such that their corresponding spaces $R'^{(j)}$ meet in a line. For a particular point of $T^{(j)}$, $R'^{(j)}$ is indeterminate whence it corresponds to an S'_k. The locus of these S'_k's is a spread $N'^{(j)n-k-3}$ of dimension $k(n-k-2)-1$ and of order $n-k-3$ since a line L' meets it in the same number of points as the corresponding curve in Σ meets $T^{(j)}$. The points of the second kind lie on a singular manifold M of dimension $k(n-k-2)-2$ which correspond to the spread M'^{n-k-3} in Σ'. Hence

(52) *The order of the Cremona transformation C is $n-k-2$; its singular manifolds in Σ are M and the $n-k-2$ spaces $T^{(j)}$. The corresponding fundamental spreads in Σ' are all of order $n-k-3$.*

Thus an $S_{k(n-k-2)-1}$ in Σ is transformed by C into a spread of order $n-k-2$ in Σ' on the $n-k-1$ singular spreads in Σ'. This again is transformed by

C^{-1} into a spread of order $(n-k-2)^2$ in Σ from which there separates the $n-k-1$ fundamental spreads of order $n-k-3$ each, leaving the original linear spread.

For the general mapping described above the choice of $T^{(j)}$ implies

$$(k+1)k(n-k-3)$$

constants and the choice of the projectivity with S_k of spaces $R^{(j)}$ upon it implies $k(k+2)$ constants. Thus the mapping depends upon

$$k(n-k-2)(n-k-3)$$

absolute constants in Σ, a number which vanishes only when $n=k+3$. Now it is very desirable that the mapping apparatus should involve no absolute constants in Σ, both in the interest of simplicity and in order that the rôle played by the absolute constants of P_n^k itself be not obscured. For these reasons the following scheme, referred to hereafter as the "canonical mapping" seems most useful. In Σ choose an origin O and a space at ∞, $u=0$; and let the $k(n-k-2)$ rectangular coördinate spaces be divided into $n-k-2$ sets of k each say $y_i^{(j)}=0$ $(i=1,\cdots,k;\ j=1,\cdots,n-k-2)$. The space $T^{(j)}$ is then chosen to be $u=y_i^{(j)}=0$ $(i=1,\cdots,k)$. Thus all the spaces $T^{(j)}$ lie in the space at infinity. Each $T^{(j)}$ is a reference space in Σ whose skew reference space is an $S_k^{(j)}$. If the unit point in Σ be projected from $T^{(j)}$ upon the thus defined unit point of $S_k^{(j)}$ we obtain thereby a basis in $S_k^{(j)}$ which can be identified with the basis in S_k itself and thus the projectivity with S_k of spaces $R^{(j)}$ on $T^{(j)}$ is established. This mapping may be described analytically as follows:

(53) *If the set P_n^k in S_k be linearly transformed so that the first $k+2$ points fall upon the reference points and the unit point and if the coördinates of the remaining points be affected with such a factor of proportionality that the last coördinate of each is u, then the other $k(n-k-2)$ coördinates of these points together with u constitute the coördinates of the canonical map P in $\Sigma_{k(n-k-2)}$ of the set P_n^k in S_k. Two distinct canonical maps are equivalent under linear transformation.*

The coördinates of the point P in Σ are $y_i^{(j)}$, u where $y_i^{(j)}$ is the i-th coördinate of the $(k+2+j)$-th point of P_n^k after the set has been prepared as described in (53). Here $i=1,\cdots,k$ and $j=1,\cdots,n-k-2$, though it is sometimes convenient to let i take the value $k+1$ in which case $y_{k+1}^{(j)}=u$ $(j=1,\cdots,n-k-2)$.

If from the *first* $k+1$ points of P_n^k in order we drop the point p_r and adjoin from the remaining points of P_n^k the point p_s, the determinant of the resulting set of $k+1$ points will be denoted by $P(r;s)$; if from the *last* $n-k-1$ points of Q_n^{n-k-2} in order we drop the point q_r and prefix from the earlier points of Q_n^{n-k-2} the point q_s, the determinant of the resulting set of $n - k - 1$ points will be denoted by $Q(r; s)$. Then

(54) *The coördinates of the canonical map P of P_n^k in terms of the coördinates of the points of P_n^k are*

$$y_i^{(j)} = P(i;\ k + 2 + j) \cdot \delta_i \cdot \epsilon_j \qquad (i = 1, \cdots, k + 1;\ j = 1, \cdots, n - k - 2),$$

where

$$\delta_i = \prod_{l=1}^{k+1} P(l;\ k + 2)/P(i;\ k + 2),$$

$$\epsilon_j = \prod_{l=1}^{n-k-2} P(k + 1;\ k + 2 + l)/P(k + 1;\ k + 2 + j).$$

For the linear transformation which converts the first $k + 2$ points of P_n^k into the canonical basis is

$$x_i' = P(i;\ x)/P(i;\ k + 2) \qquad \text{or} \qquad x_i' = P(i;\ x) \cdot \delta_i \qquad (i = 1, \cdots, k + 1).$$

Then the last coördinate of the remaining points is

$$x_{k+1}'^{j} = P(k + 1;\ k + 2 + j) \cdot \delta_{k+1} \quad (j = 1, \cdots, n - k - 2).$$

These are made equal by multiplication with ϵ_j respectively.

If the set P_n^k has been transformed into the canonical form and if all of the non-vanishing coördinates of the base points have been taken to be unity which we shall call the "prepared" form of P_n^k, then the determinants formed from them satisfy the following numerical relations:

$$P(k + 1;\ k + 2) = - P(k;\ k + 2) = P(k - 1;\ k + 2) = \cdots$$
$$= (-1)^k P(1;\ k + 2) = 1,$$

(55) $$P(k + 1;\ k + 3) = P(k + 1;\ k + 4) = P(k + 1;\ k + 5) = \cdots$$
$$= P(k + 1;\ n) = u,$$
$$P(i;\ k + 2 + j) = (-1)^{k+i+1} y_i^{(j)} \qquad (i = 1, \cdots, k \text{ or } k + 1;\ j = 1, \cdots, n - k - 2).$$

The associated sets P_n^k and Q_n^{n-k-2} map into points P and P' in spaces $\Sigma_{k(n-k-2)}$ and $\Sigma'_{k(n-k-2)}$ respectively. Let us determine the Cremona transformation from P' to P. Let the sets be given in prepared form in which case the identity (2) is

$$\sum_{r=1}^{n} (up_r) \cdot \lambda_r (vq_r) = 0.$$

The determinants formed from the points p are proportional to those formed from the points q, the latter each being multiplied by the product of the λ's corresponding to the points of the determinant. Using a notation for such a product similar to the notation for the determinant we have

$$P(k+1;\, k+2) = \epsilon \cdot \lambda(k+2;\, k+1) \cdot Q(k+2;\, k+1),$$
$$P(k;\, k+2) = -\epsilon \cdot \lambda(k+2;\, k) \cdot Q(k+2;\, k),$$
$$\cdots\cdots\cdots\cdots$$
$$P(1;\, k+2) = (-1)^k \epsilon \cdot \lambda(k+2;\, 1) \cdot Q(k+2;\, 1).$$

Using the first of the relations (55) and the fact that

$$\lambda(k+2;\, k+1) : \lambda(k+2;\, k) : \cdots : \lambda(k+2;\, 1) = \lambda_{k+1} : \lambda_k : \cdots : \lambda_1$$

we find that

$$(a)\quad \lambda_1 : \lambda_2 : \cdots : \lambda_{k+1} = \frac{1}{Q(k+2;\, 1)} : \frac{1}{Q(k+2;\, 2)} : \cdots : \frac{1}{Q(k+2:\, k+1)}.$$

If we use similarly the second of relations (55) we get

$$(b)\quad \begin{aligned} u &= P(k+1;\, k+3) = -\epsilon \cdot \lambda(k+3;\, k+1)Q(k+3;\, k+1),\\ u &= P(k+1;\, k+4) = \epsilon \cdot \lambda(k+4;\, k+1)Q(k+4;\, k+1),\\ &\cdots\cdots\cdots\cdots\\ u &= P(k+1;\, n) = (-1)^{n-k-2} \epsilon \cdot \lambda(n;\, k+1)Q(n;\, k+1),\end{aligned}$$

whence

$$(c)\quad \lambda_{k+3} : \lambda_{k+4} : \cdots : \lambda_n = Q(k+3;\, k+1) : (-1)Q(k+4;\, k+1) : \cdots : (-1)^{n-k-3} Q(n;\, k+1).$$

Finally from the last of relations (55) we have

$$(d)\quad (-1)^{k+i+1} y_i^{(j)} = P(i;\, k+2+j) = (-1)^{k+j+i+1} \epsilon \cdot \lambda(k+2+j;\, i)Q(k+2+j;\, i).$$

In (*d*) and (*b*) the products λ all contain λ_{k+2} to the first degree and this may be rejected as a factor of proportionality. They also each contain one factor from the set $\lambda_1, \cdots, \lambda_{k+1}$ and $n-k-3$ factors from the set $\lambda_{k+3}, \cdots, \lambda_n$ so that the factors of proportionality in (*a*) and (*c*) are not material. Dividing by the product of $\lambda_{k+3}, \cdots, \lambda_n$ we have finally that

(56) *The coördinates of the map P of the set P_n^k in terms of the coördinates of the map P′ of the associated set Q_k^{n-k-2} are*

$$y_i^{(j)} = (-1)^{(j)} Q(k+2+j;\, i)/Q(k+2;\, i)Q(k+2+j;\, k+1)$$
$$(i = 1, \cdots, k+1;\ j = 1, \cdots, n-k-2).$$

If this be cleared of fractions by multiplying by $\prod_i Q(k+2;\, i)$ and

$$\prod_j Q(k+2+j;\, k+1),$$

the right-hand members are of degree 1 in each of $q_1, \cdots, q_k$; of degree $n-k-2$ in each of q_{k+1}, q_{k+2}; and of degree $n-3$ in each of $q_{k+3}, \cdots, q_n$. The order of the transformation depends upon the norm $N_{n,k}$. If this be zero and $n = 2k+2$, the coördinates of $q_1, \cdots, q_{k+2}$ are numerical and the apparent order is $(2k-1)k$. But if a determinant Q contains neither q_{k+2} nor q_{k+1} it must contain the factor v. Thus $\Pi_i Q(k+2; i)$ contains the factor v^k while each denominator contains the factor v to at most the first power and the apparent order is reduced by $k-1$. Hence

(57) *The order in* Σ_{k^2} *of the involutory Cremona transformation between the maps of the associated sets* P_{2k+2}^k *and* Q_{2k+2}^k *is* $k^2 + (k-1)^2$.

A simple form of two associated sets which shows the reciprocity between the two is

$$\begin{array}{llllll|llll}
(1) & 1, & 0, & \cdots, & 0, & 0 & y_1^{(n-k-2)}, & y_1^{(n-k-3)}, & \cdots, & y_1^{(1)}, u \\
(2) & 0, & 1, & \cdots, & 0, & 0 & y_2^{(n-k-2)}, & y_2^{(n-k-3)}, & \cdots, & y_2^{(1)}, u \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(k) & 0, & 0, & \cdots, & 1, & 0 & y_k^{(n-k-2)}, & y_k^{(n-k-3)}, & \cdots, & y_k^{(1)}, u \\
(k+1) & 0, & 0, & \cdots, & 0, & 1 & 1, & 1, & \cdots, & 1, \; 1 \\
(k+2) & 1, & 1, & \cdots, & 1, & 1 & 0, & 0, & \cdots, & 0, \; 1 \\
(k+3) & y_1^{(1)}, & y_2^{(1)}, & \cdots, & y_k^{(1)}, & u & 0, & 0, & \cdots, & 1, \; 0 \\
(k+4) & y_k^{(2)}, & y_k^{(2)}, & \cdots, & y_k^{(2)}, & u & 0, & 0, & \cdots, & 0, \; 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(n-1) & y_1^{(n-k-3)}, & y_2^{(n-k-3)}, & \cdots, & y_k^{(n-k-3)}, & u & 0, & 1, & \cdots, & 0, \; 0 \\
(n) & y_1^{(n-k-2)}, & y_2^{(n-k-2)}, & \cdots, & y_k^{(n-k-2)}, & u & 1, & 0, & \cdots, & 0, \; 0
\end{array}$$

One verifies at once that these sets satisfy an identity

$$\sum_{l=1}^{n} (up_l) \cdot \lambda_l (vq_l) \equiv 0$$

for $\lambda_1 = \cdots = \lambda_k = 1, \lambda_{k+1} = \lambda_{k+2} = -u, \lambda_{k+3} = \cdots = \lambda_n = -1$. They are not however simultaneously in the prepared form because of the inverted order of Q_n^{n-k-2} with reference to P_n^k.

7. The Cremona group of order $n!$ in $\Sigma_{k(n-k-2)}$ determined by P_n^k

In this paragraph we first consider the effect of changing the order of the points in P_n^k. If P is the map of P_n^k in $\Sigma_{k(n-k-2)}$ and if the permutation π is applied to the points of P_n^k, the map P' in Σ of $P_n'^k$, the permuted set, will be different from P unless P_n^k is self-projective according to the permutation π. But this is not in general the case when $n > k+2$. If P be given, the prepared form of P_n^k is known (54). If π be applied to this prepared form and the resulting point set be prepared by proper linear transformation in S_k, then the point P' is determined. All of the operations involved in this process are rational whence P' and P correspond in a Cremona transformation of Σ. The $n!$ permutations π thus lead to $n!$ points P' including P itself. These $n!$ points have the property that their $n!$ sets $P_n'^k$ can all be superposed, by proper linear transformation in S_k, upon any definite one say P_n^k in some order. Thus P_n^k may be any one of the $n!$ sets and

(58) *The $n!$ points P' are a set of conjugate points under a Cremona group, $G_{n!}^k$, of order $n!$ in $\Sigma_{k(n-k-2)}$, which is isomorphic with the permutations π of the points in P_n^k.*

A set of generating permutations which leads to convenient generators of the Cremona group is made up of the transpositions $(l, l+1)$ $(l = 1, \cdots, n-1)$. These values of l divide into four sets, $1, \cdots, k$; $k+1$; $k+2$; $k+3, \cdots, n$, according to the form of the corresponding Cremona transformation. If the points p_l, p_{l+1} of the first set be interchanged, the linear transformation in S_k which restores the original basis is (x_l, x_{l+1}). If we note the effect of this upon the remaining $n-k-2$ points we find that the corresponding transformation in Σ is

$$(59)\qquad \begin{aligned} &(l, l+1): \quad y_l'^{(j)} = y_{l+1}^{(j)}, \quad y_i'^{(j)} = y_i^{(j)} \\ &\qquad\qquad\qquad (i = 1, \cdots, k \text{ but } i \neq l, l+1), \\ &(l = 1, \cdots, k-1), \quad y_{l+1}'^{(j)} = y_l^{(j)} \quad u' = u \\ &\qquad\qquad\qquad (j = 1, \cdots, n-k-2). \end{aligned}$$

The transposition $(k, k+1)$ interchanges the last two reference points. The linear transformation in S_k which restores them is (x_k, x_{k+1}). This destroys the equality of the last coördinates in the prepared form so that each of the new points $p_{k+3}, \cdots, p_n$ must be affected by a factor of proportionality which is the product of the last coördinates of the remaining ones. Hence the Cremona transformation in Σ is

$$(60)\qquad \begin{aligned} &(k, k+1): \quad y_m'^{(j)} = r_j\, y_m^{(j)}, \quad y_k'^{(j)} = r_j\, u, \quad u' = r, \\ &(m = 1, \cdots, k-1), \quad r = \prod_{m=1}^{n-k-2} y_k^{(m)}, \quad r_j = r/y_k^{(j)}. \end{aligned}$$

This transformation is of order $n-k-2$.

The transposition $(k+1, k+2)$ interchanges the last reference point and the unit point. The linear transformation in S_k which restores them is $x_l' = u - x_l$ $(l = 1, \cdots, k)$, $u' = u$. This gives rise to a linear transformation in Σ,

$$(61)\qquad (k+1, k+2): \quad y_i'^{(j)} = u - y_i^{(j)}, \quad u' = u \quad (i = 1, \cdots, k;\ j = 1, \cdots, n-k-2).$$

The transposition $(k+2, k+3)$ interchanges the unit point and the point p_{k+3} whose coördinates are $x_l = y_l^{(1)}$ $(l = 1, \cdots, k)$ and $x_{k+1} = u$. The linear transformation which leaves the reference basis unaltered and transforms p_{k+3} into the unit point is $x_l' = x_l/y_l^{(1)}$, $x_{k+1}' = x_{k+1}/u$. It transforms the unit point into $1/y_1^{(1)}, \cdots, 1/y_k^{(1)}, 1/u$, and the point p_{k+2+j} into $y_1^{(j)}/y_1^{(1)}, \cdots, y_k^{(j)}/y_k^{(1)}, 1$ $(j = 2, \cdots, n-k-2)$. We must therefore multiply the coördinates of p_{k+3}' by u and then clear the coördinates of fractions. The resulting Cremona transformation is

$$(62)\quad \begin{aligned} &(k+2, k+3): \quad y_l'^{(j)} = r_l y_l^{(j)}, \quad y_l'^{(1)} = u r_l, \quad u' = r \\ &(j = 2, \cdots, n-k-2;\ l = 1, \cdots, k), \quad r = \prod_{l=1}^{k} y_l^{(1)},\ r_l = r/y_l^{(1)}. \end{aligned}$$

This transformation is of order k.

For the last type of transposition the coördinates of the two points of P_n^k are merely interchanged so that the corresponding transformation in Σ is linear,

$$(63)\quad \begin{aligned} &(l, l+1): \quad y_i'^{(l)} = y_i^{(l+1)}, \quad y_i'^{(l+1)} = y_i^{(l)}, \quad y_i'^{(j)} = y_i^{(j)}, \quad u' = u \\ &(l = k+3, \cdots, n-1;\ i = 1, \cdots, k;\ j = 1, \cdots, n-k-2 \text{ but } j \neq l, l+1). \end{aligned}$$

(64) *The Cremona* $G_{n!}^k$ *in* $\Sigma_{k(n-k-2)}$ *isomorphic with the* $g_{n!}$ *of permutations of* P_n^k *is generated by the elements* (59), $\cdots$, (63).

The $G_{n!}$ contains a collineation subgroup of order $2!\,k!\,(n-k-2)!$ generated by (59), (61), and (63). This is the direct product of the permutable groups of orders $k!$, $(n-k-2)!$, and $2!$ which correspond to the permutations of the columns, the permutations of the rows of the matrix $||y_i^{(j)}||$, and the involutory collineation which interchanges the origin and unit point of Σ leaving the other reference points unchanged. Thus the two points of the base in Σ form one conjugate set and the remaining base points form another conjugate set under collineations of $G_{n!}$. If $k = 1$ $[n-k-2 = 1]$ the transformation (62) [(60)] is also a collineation so that all the transpositions except the first [last] lead to collineations which generate a collineation subgroup of order $(n-1)!$.

The remainder of this paragraph is devoted to the discussion of the invariants of $G_{n!}$ and their relation to the invariants of P_n^k as defined in § 3.

An invariant of a Cremona group is a form which is unaltered by each element of the group to within a factor composed of the fundamental spreads associated with the singular spreads of the particular element. Let

$$f_\rho(y_i^{(j)}, u) = \sum \Pi (y_i^{(j)})^{a_{ij}} u^\epsilon$$

be an invariant of $G_{n!}$ of order ρ. Let us consider the effect of the transformation $(k, k+1)$ upon f_ρ. This transformation has the $n-k-2$

singular manifolds $\Sigma^{(l)}$ $(l = 1, \cdots, n-k-2)$ defined by

$$y_i^{(j)} = u = 0 \quad (j = 1, \cdots, n-k-2;\ j \neq l;\ i = 1, \cdots, k),$$

whose corresponding fundamental spreads are $y_k^{(l)} = 0$; and the further singular manifold S defined by $y_k^{(j)} = 0$ $(j = 1, \cdots, n-k-2)$, whose fundamental spread is $u = 0$. If f_ρ contains a particular $\Sigma^{(l)}$ s times it must contain each $\Sigma^{(l)}$ s times since it is invariant under collineations which interchange the $\Sigma^{(l)}$. Suppose that it contains S also r times. Then it is transformed by $(k, k+1)$ into a spread of order $\rho(n-k-2)$ from which the fundamental spreads $y_k^{(j)}$ factor s times each and the fundamental spread u factors r times leaving the invariant factor f_ρ, i. e.,

$$s(n-k-2) + r = (n-k-3)\rho.$$

For a particular term $\pi(y_i^{(j)})^{\alpha_{i,j}} u^\epsilon$ let

$$\sum_{i,j} \alpha_{i,j} = \alpha = \rho - \epsilon$$

and let

$$\sum_{i=1}^{k} \alpha_{i,j} = \alpha_j.$$

Then since each term must contain $\Sigma^{(j)}$ at least s times we have that

$$\alpha + \epsilon - \alpha_1 = s + s_1, \qquad \alpha + \epsilon - \alpha_2 = s + s_2, \qquad \cdots,$$

$$\alpha + \epsilon - \alpha_{n-k-2} = s + s_{n-k-2},$$

where the s_j's are zero or positive. By addition,

$$(n-k-3)\alpha + (n-k-2)\epsilon = (n-k-2)s + \sum_j s_j \quad \text{or} \quad r + \epsilon = \sum_j s_j.$$

If then f_ρ be multiplied by u^r we get an invariant $f'_{\rho'}$ of order $\rho' = \rho + r$ which acquires under $(k, k+1)$ the factor

$$\prod_{j=1}^{n-k-2} (y_k^{(j)})^{s+r}.$$

In each term $\epsilon' = r + \epsilon = \Sigma s_j$ is such that $u^{\epsilon'}$ can be separated into the product $u^{s_1} u^{s_2} \cdots u^{s_{n-k-2}}$ where $\alpha_j + s_j = \alpha + \epsilon - s$. But $\alpha_j + s_j$ is the degree of f'_ρ in the coördinates of the $(k+2+j)$th point if u in u^{s_j} be regarded as the $(k+1)$th coördinate of this point. Hence

(65) *Every invariant of $G_{n!}$ after multiplication by a proper power of u can be expressed as a form homogeneous and of the same degree ν in the coördinates of the $n-k-2$ points p_{k+2+j} $(j = 1, \cdots, n-k-2)$. The index of this power is in general the multiplicity of the singular manifold S on the invariant spread.*

The case of exception to the above value of the index is mentioned after (68).

We shall suppose hereafter that all invariants, i, of the Cremona group have been normalized as above by introducing a proper power of u. By using the formulæ of (54) the invariant i can be expressed in terms of the coördinates of the points of the general point set P_n^k. This expression must factor into an effective factor I depending upon the set and into an extraneous factor E depending upon the choice of the base. If for example the separation of $u^{\epsilon'}$ has been carried out as above the factors ϵ_j^ν from (54) will form a part of E and the factor I will then be of degree ν in the coördinates of each of the last $n-k-2$ points. Thus the vanishing of i implies that if all but one of the set P_n^k be fixed in S_k the remaining one must lie on a spread of order ν in S_k. Since as an invariant of $G_{n!}$ i has the same meaning irrespective of the permutations of the points, I must contain all the points to the same degree ν and contain them symmetrically. Moreover since the $y_i^{(j)}$ and u are invariants in the projective sense of P_n^k, I must be expressible in terms of determinants formed from the points of P_n^k. Hence I is an invariant of P_n^k as defined in § 3 and its degree ν must be such that $n\nu$ is divisible by $k+1$. All that has been said of spreads invariant under $G_{n!}$ applies equally well to invariant linear systems of spreads whence

(66) *Every normalized invariant spread or invariant linear system of spreads of $G_{n!}$ has in Σ the order $\nu(n-k-2)$ where $n\nu$ is divisible by $k+1$. When expressed in terms of the general set P_n^k it gives rise, to within an extraneous factor depending upon the base selected from P_n^k, to an invariant or linear system of irrational invariants of P_n^k of degree ν.*

That the degree ν of an invariant or irrational invariant is subject to no other condition than that given above is evident from the construction of the following simplest linear system of irrational invariants. Let D be the greatest common divisor of n and $k+1$, i. e., let $n = Dm$ and $k+1 = Dl$, where l and m are relatively prime. Arrange the n points of P_n^k in m sets of D each and from l of the m sets form a determinant of order $k+1$. By permuting the l sets cyclically through the m sets we get m determinants whose product is an irrational invariant of P_n^k of degree l, the lowest degree consistent with (66). The irrational invariants of this type give rise in Σ to an invariant linear system of spreads whose normalized order is $l(n-k-2)$. Hence

(67) *The linear system of spreads of lowest order in $\Sigma_{k(n-k-2)}$ invariant under $G_{n!}$ has the normalized order $l(n-k-2)$ where $k+1 = Dl$, D being the greatest common divisor of n and $k+1$.*

So far as the invariance of a spread in Σ under $G_{n!}$ is concerned the normalizing factor u^r is extraneous. Its value in the case above and indeed in general can be determined. If $N_{n,k} = n-2k-2 \leqq 0$, P_n^k can be arranged in the order $p_{k+1}, p_1, \cdots, p_k, p_{k+2}, \cdots, p_n$. Then the product of n determinants, the first formed from the first $k+1$ points and the others formed from points succeeding these cyclically, is such that each determinant contains either p_{k+1} or p_{k+2} and therefore the corresponding spread in Σ does not contain the factor u. This product consists of D irrational invariants of the type (67) so that for this simplest linear system no normalizing factor is required. The simplest linear system for the associated Q_n^{n-k-2} is obtained by using the complementary determinants. But from the associated sets given on p. 182, § 6, it is clear that the values of the complementary determinants are the same to within factors u^s and the two linear systems coincide. But the normalized order of the second system as given by (67) is kl' where $n-k-1 = D'l'$, D' being the G.C.D. of n and $n-k-1$. Hence $D' = D$ and $l' = (n-k-1)/D$. Thus the difference between the actual order of the first system, $(n-k-2)(k+1)/D$ and the normalized order of the second, $k(n-k-1)/D$ is $r = (2k+2-n)/D = N_{n,\,n-k-2}/D$.

(68) *The normalized order of the invariant linear system* (67) *exceeds the actual order by r if $N_{n,\,k} = Dr > 0$.*

Since the order of any invariant is presumably a multiple of this lowest order the order of the normalizing factor is determined in general. However as the order of an invariant increases and a number of linearly independent invariants begin to appear, proper linear combinations can be formed so that u appears as a factor independently of the normalizing factor. This occurs when the invariant spread contains the singular manifold S of $(k, k+1)$ a larger number of times than is customary. Naturally in selecting the members of a complete system it would be convenient to choose those invariants which contain the highest possible power of u as a factor. This device is used in the discussion of P_6^2 in Part III of this paper.

The $G_{n!}$ associated with P_n^k appears as a natural generalization of Moore's "cross-ratio" group and we might expect to find in S_k some analogue of the cross-ratio of 4 points in S_1. If the points $p_3, \cdots, p_{k+2}$ lie in the S_{k-1}, $(\xi x) = (p_3, \cdots, p_{k+2}, x)$ and if the points $p_{k+3}, \cdots, p_{2k+2}$ lie in the S_{k-1}, $(\eta x) = (p_{k+3}, \cdots, p_{2k+2}, x)$, then

$$(69) \quad C = \frac{(\xi p_1)(\eta p_2)}{(\xi p_2)(\eta p_1)} = \frac{(3, 4, \cdots, k+2, 1)(k+3, \cdots, 2k+2, 2)}{(3, 4, \cdots, k+2, 2)(k+3, \cdots, 2k+2, 1)}.$$

is defined to be *a cross-ratio of the $2k+2$ points in S_k*; or more precisely *the cross-ratio of the pair* 1, 2 *with respect to the sets* 3, $\cdots$, $k+2$ *and* $k+3, \cdots,$ $2k+2$. It is in fact the ordinary cross-ratio on the line $\overline{12}$ of the points 1, 2 with respect to the points where the line meets the two S_{k-1}'s. Its self-dual character is obvious. Moreover it is an absolute irrational invariant of the $2k+2$ points. Analogous irrational invariants of the same degree can be formed; for example when $k=3$ we have $\frac{(1234)(5678)}{(1256)(3478)}$. Since these irrational invariants can be expressed by means of the above cross-ratios [thus $\frac{(1234)(5678)}{(1256)(3478)} = \frac{(1234)(5678)}{(1235)(4678)}; \frac{(1235)(4678)}{(1256)(3478)}$] and since they apparently are not binary cross-ratios determined by the $2k+2$ points it seems better to restrict the term as in (69).

The cross-ratios are formed from the ratios of members of the simplest invariant linear system determined by P_{2k+2}^k. The dimension of this linear system can be determined. For $k = 1$ we have the three invariants $(12)(34)$, $(13)(42)$, $(14)(23)$, of which two are independent. For $k = 2$ we have ten invariants like $(123)(456)$ of which according to (34) five are independent. For $k = 3$, there are 35 invariants like $(1234)(5678)$ of which 14 are linearly independent. For each product in which 7 and 8 do not appear in the same factor is paired with another; e. g. $(ijk7)(lmn8)$ is paired with $(ijk8)(lmn7)$. The difference of the two can be expressed in terms of invariants like $(ijkl)(mn78)$. The sum of the two is the polar of the point pair 7, 8 as to the quadric $(ijkx)(lmnx)$. From the principle of translation just five of these quadrics are linearly independent. There are 15 invariants like $(ijkl)(mn78)$ but they are connected by 6 determinant identities. Moreover these 6 identities are linearly independent. For if the identities were linearly related then in this relation all terms containing the triad $v = \overline{ijk}$ within a determinant would cancel. But the identities contain $v = \overline{ijk}$ and $w = \overline{78}$ as follows:

$$(vl)(mnw) - (vm)(lnw) \cdots = 0, \qquad (vl)(mnw) - (vn)(lmw) \cdots = 0,$$
$$(vm)(lnw) - (vn)(lmw) \cdots = 0,$$

and these combinations are linearly independent. The identities then reduce the number 15 to 9 which with the 5 already obtained make up the number 14 of linearly independent invariants. The numbers 2, 5, 14 for $k = 1, 2, 3$ are given by $\binom{2k+2}{k+1}\frac{1}{k+2}$. Assuming this to be the true value up to k let us prove that it is true for k. Selecting as above two points we find (for the value $k-1$) $\binom{2k}{k}\frac{1}{k+1}$ independent polar forms leaving $\binom{2k}{k-1}$ products connected by $\binom{k+1}{k-2}$ identities. That the identities are independent is proven as above by taking k of the points to make up v and $k-3$ together with the two originally selected to make up w. Thus the number of linearly independent products is $\binom{2k}{k}\frac{1}{k+1} + \binom{2k}{k-1} - \binom{2k}{k-2} = \binom{2k+2}{k+1}\frac{1}{k+2}$. Hence

(70) *There are* $\binom{2k+2}{k+1}\frac{1}{k+2}$ *linearly independent determinant products of the type* $(1, 2, \cdots, k+1)(k+2, \cdots, 2k+2)$ *that can be formed from* $2k+2$ *points in* S_k. *The invariant linear system of lowest order,* (67), *under* $G_{(2k+2)!}$ *in* Σ_{k^2} *has the dimension* $\binom{2k+2}{k+1}\frac{1}{k+2} - 1$ *and the order* k.

Identities of higher degree connecting these products must exist when $k > 2$ since the dimension of the linear system is larger than the number of absolute constants in P_{2k+2}^k.

The cross-ratios of the set P^k_{2k+2} do not uniquely define the set when $k > 1$ since they take the same value for the associated set Q^k_{2k+2}. To within this ambiguity however they undoubtedly suffice to determine the set projectively.

If for P^k_n, $N_{n,k} > 0$ we can define a cross-ratio of the set P^k_n to be a cross-ratio of any $2k + 2$ points of the set. Such a cross-ratio is an incomplete irrational absolute invariant of P^k_n. If then all the cross-ratios of P^k_n be given (subject of course to the algebraic relations connecting them) the set can be uniquely determined. For the ambiguity inherent in the determination of $2k + 2$ points of P^k_n must be resolved when the conditions implied by the further cross-ratios are applied.

If for P^k_n, $N_{n,k} < 0$, then for the associated set Q^{n-k-2}_n, $N_{n,n-k-2} > 0$ and we can define the cross-ratios of P^k_n to be those of its associated set.

8. The form problem of P^1_5 and P^2_5

For $n = 5$ we have but two point sets, the associated P^1_5 and P^2_5. If we arrange their prepared forms in the following orders:

(1)	$1, 0$	y, x, u
(2)	$0, 1$	$1, 1, 1$
(3)	$1, 1$	$0, 0, 1$
(4)	x, u	$0, 1, 0$
(5)	y, u	$1, 0, 0$

the Cremona $G_{5!}$ in the space Σ_2 (variables x, y, u) has according to § 7 the same form for both sets,

(71)

	(12)	(23)	(34)	(45)
x'	$= uy$	$= u - x$	$= u$	$= y$
y'	$= ux$	$= u - y$	$= y$	$= x$
u'	$= xy$	$= u$	$= x$	$= u$

The simplest invariant linear system is derived in the case of P^1_5 from products like (12)(23)(34)(45)(51), and in the case of P^2_5 from products like (345)(451)(512)(123)(234). In the first case according to (68) the products all contain the normalizing factor u. If this be removed the linear system in both cases is the system of ∞^5 cubic curves on the reference and unit points in Σ_2.

The invariants of $G_{5!}$ are merely the invariants of the binary quintic determined by P^1_5. They have been given from this point of view in $C1$ where also the form-problem of $G_{5!}$ has been set forth and solved by means of Klein's A-problem and ultimately in terms of the ikosahedral irrationality. The solution there given is both direct and explicit. It may be worth while to point out here the rôle played in this solution by the above linear system of irrational invariants.

Instead of using the 12 cyclic irrational invariants let us replace them by the 10 irrational invariants like $(ij)^2(kl)(km)(lm)$ in terms of which the cyclic invariants can be expressed. They are connected by 5 linear relations of the type

$$(72)\quad (ij)^2(kl)(km)(lm) - (ik)^2(jl)(jm)(lm) + (il)^2(jk)(jm)(km) - (im)^2(jk)(jl)(kl) = 0.$$

The 5 relations are themselves linearly related so that there are 6 linearly independent invariants.

If we take a point set P_5^3 in S_3 which forms a base we can use a superfluous coördinate and write

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0.$$

A line π determined by points x and y has 10 coördinates connected by a set of 5 linear relations similar to (72),

$$(73)\qquad \pi_{ij} + \pi_{ik} + \pi_{il} + \pi_{im} = 0 \qquad (i = 1, \cdots, 5;\quad \pi_{ik} = -\pi_{ki}),$$

which also are linearly related so that 6 of the coördinates are linearly independent. A linear complex is given by an equation

$$(74)\qquad \sum_{i,k} a_{ik}\,\pi_{ik} = 0 \qquad (i, k = 1, \cdots, 5;\ i \neq k;\ a_{ik} = -a_{ki}).$$

Since by virtue of (73) this complex can also be written as

$$\sum_{i,k} a'_{ik}\,\pi_{ik} = \sum_{i,k} (a_{ik} + \lambda_i)\,\pi_{ik} = 0,$$

the λ's can be so chosen that

$$(75)\qquad a'_{ij} + a'_{ik} + a'_{il} + a'_{im} = 0 \qquad (i = 1, \cdots, 5).$$

Thus only 6 of the 10 complex coördinates are independent.

(76) *If we set* $a'_{ij} = (-1)^\epsilon (ij)^2(kl)(km)(kn)$ $(i < j)$, *where* ϵ *is odd or even according as the permutation ijklm is odd or even, then the point* x, y, u *of* Σ_2 *determines a linear complex in* S_3 *whose transformations by the collineation* $G_{5!}$ *of* P_5^3 *are isomorphic with the transformations of* x, y, u *in* Σ_2 *by the Cremona* $G_{5!}$.

This is the linear complex utilized in $C1$ to obtain the transition from the given form problem to Klein's A problem. That it is the linear complex determined by the twisted cubic on P_5^3 whose parameters are given by P_5^1 one can verify by a direct calculation. The analytic conditions which differentiate it from the general linear complex arise from the fact that the irrational invariants employed satisfy five quadratic identities. The existence of these identities is evident from the fact that with each cyclic invariant,

$$\alpha = (12)(23)(34)(45)(51),$$

there is associated another,

$$\alpha' = (13)(35)(52)(24)(41),$$

such that $\alpha\alpha' = -\sqrt{D}$ where $\sqrt{D} = \prod (ik) \quad (i < k)$. Thus the 12 cyclic invariants are connected by 5 quadratic relations of the form

$$\alpha\alpha' = \beta\beta' = \cdots = \zeta\zeta'.$$

For the ten a_{ij}'s there must exist five equivalent relations. The corresponding fact for the cubic curves in Σ_2 on a base is that they map the Σ_2 upon a quintic 2-way in S_5 which is the complete intersection of 5 quadrics.

In the solution of the quintic given in $C1$ the use of the Cremona group and its invariant system of cubic curves might have been avoided entirely by replacing the curves by the irrational invariants a'_{ij} of the quintic. Such a process however would be merely a circumlocution which would obscure the real nature of the case.

9. The parametric expression of a self-associated P_6^2 by means of theta-modular functions

Using the methods of finite geometry as set forth in $C3$ we identify the 15 rational period characteristics (Per. Char.) with denominator 2 of the theta function in two variables with the coördinates $\binom{x_1\ x_2}{x_3\ x_4}$ of a point in the finite geometry modulo 2 in S_3. These points can be arranged in a basis configuration [$C3$, p. 271]

	B_1	B_2	B_3	B_4	B_5	B_6
B_1		$\begin{pmatrix}0&1\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}0&0\\1&1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$
B_2	$\begin{pmatrix}0&1\\0&0\end{pmatrix}$		$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&1\end{pmatrix}$	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$
B_3	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$		$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0\\1&0\end{pmatrix}$
B_4	$\begin{pmatrix}0&0\\1&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$		$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\1&0\end{pmatrix}$
B_5	$\begin{pmatrix}0&1\\0&1\end{pmatrix}$	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$		$\begin{pmatrix}1&1\\0&0\end{pmatrix}$
B_6	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}0&0\\1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\0&0\end{pmatrix}$	

which is made up of 6 bases, each being the 5 points in a row or in a column. Since each point is in two bases and any two bases have a common point, a point can be denoted by P_{ik} $(i, k = 1, \cdots, 6; i \neq k)$ if it belongs to the bases B_i and B_k.

The 16 odd and even theta functions are identified with 16 quadrics in the finite geometry. The function $\vartheta(u) = \vartheta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](u)$ is identified with the quadric $Q = x_1 x_3 + x_2 x_4 = 0$, and the function $\vartheta\left[\begin{smallmatrix}ij\\kl\end{smallmatrix}\right](u)$ with the quadric obtained by adding to Q the square of the polar plane of the point $\left(\begin{smallmatrix}ij\\kl\end{smallmatrix}\right)$ as to Q. These quadrics divide into 6 O quadrics Q_i $(i = 1, \cdots, 6)$ such that Q_i contains the five points P_{ik} of the base B_i; and into 10 E quadrics $Q_{ijk} = Q_{lmn}$ such that Q_{ijk} contains the 9 points P_{il}. A similar subscript notation is used for the corresponding theta functions. An even [odd] theta function does not [does] or does [does not] vanish for a given Per. Char. according as the corresponding quadric does or does not contain the corresponding point.

Any five theta squares are connected by a linear relation with constant coefficients. Let us find the relation connecting $\vartheta^2_{ijk}(u)$, $\vartheta^2_{ijl}(u)$, $\vartheta^2_{ijm}(u)$, $\vartheta^2_{ijn}(u)$, and say $\vartheta^2_k(u)$. If the Per. Char., P_{ij}, be substituted for u, the first four squares vanish while the last does not whence it cannot appear in the relation. Let us assume the relation to be

$$a_{ijk}\,\vartheta^2_{ijk}(u) + a_{ijl}\,\vartheta^2_{ijl}(u) + a_{ijm}\,\vartheta^2_{ijm}(u) + a_{ijn}\,\vartheta^2_{ijn}(u) = 0.$$

To determine the constants let u be P_{kl}, P_{km}, P_{kn} in turn. Then

$$a_{ijk}\,\vartheta^2_{ijk}(P_{kl}) + a_{ijl}\,\vartheta^2_{ijl}(P_{kl}) = 0,$$
$$a_{ijk}\,\vartheta^2_{ijk}(P_{km}) + a_{ijm}\,\vartheta^2_{ijm}(P_{km}) = 0,$$
$$a_{ijk}\,\vartheta^2_{ijk}(P_{kn}) + a_{ijn}\,\vartheta^2_{ijn}(P_{kn}) = 0.$$

If we use the formula*

$$\vartheta_{[\epsilon]_2}((u + \{\eta\}_2)) = e^{-\sum_{\mu=1}^{\mu=p}\sum_{\mu'=1}^{\mu'=p} a_{\mu\mu'}\frac{\eta_\mu\eta_{\mu'}}{4} - \sum_{\mu=1}^{\mu=p}\eta_\mu\left(u_\mu + \frac{\epsilon'_\mu + \eta'_\mu}{2}\pi i\right)}\vartheta_{[\epsilon+\eta]_2}((u)),$$

we find for $u = 0$ that

$$\vartheta^2_{[\epsilon]_2}((\{\eta\}_2)) = f_\eta \cdot e^{-\sum_{\mu=1}^{\mu=p}\eta_\mu\epsilon'_\mu}\,\vartheta^2_{[\epsilon+\eta]_2},$$

where $\vartheta_{ijk} = \vartheta_{ijk}(u)_{u=0}$ and f_η depends only on the Per. Char. $\{\eta\}_2$. Thus to within the factor f_η which disappears from the above equations we can set

$$\vartheta^2_{ijk}(P_{kl}) = \epsilon_{ijk,\,kl}\,\vartheta^2_{ijl},$$

where $\epsilon_{ijk,\,kl}$ is 1 or -1 according as $a\gamma + b\delta$ is congruent modulo 2 to 0 or 1

* Krazer, *Lehrbuch der Thetafunktionen*, p. 240, VII.

when $P_{kl} = \binom{ab}{cd}$ and $\vartheta_{ijk}(u) = \vartheta\left[\begin{smallmatrix}\alpha\beta\\ \gamma\delta\end{smallmatrix}\right](u)$. Hence to within sign $a_{ijk}, \cdots$, are proportional to $\vartheta^2_{ijk}, \cdots$. The signs of a_{ijk} and a_{ijl} are like or unlike according as $\epsilon_{ijk,\,kl}$ and $\epsilon_{ijl,\,kl}$ are like or unlike. But they are like or unlike according as their sum is even or odd. Since the sum of the characteristics ijk and ijl is the characteristic of P_{kl} we have only to test whether $ac + bd \equiv \{^0_1$ mod. 2, i. e., to test whether P_{kl} is or is not on the quadric $Q_{135} = Q_{246}$ associated with $\vartheta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right]$, i. e., whether $k + l$ is even or odd. Hence

(77) *The squares of the ten even thetas are connected by the* 15 *linear relations*

$$(-1)^k \vartheta^2_{ijk}\vartheta^2_{ijk}(u) + (-1)^l \vartheta^2_{ijl}\vartheta^2_{ijl}(u) + (-1)^m \vartheta^2_{ijm}\vartheta^2_{ijm}(u) + (-1)^n \vartheta^2_{ijn}\vartheta^2_{ijn}(u) = 0.$$

The ten determinant products formed from six points also are connected by (15) relations of the same kind. In any such identity two terms are obtained the one from the other by interchanging two letters and changing the sign; e. g., the pair of terms, $(-k-)(-l-) - (-l-)(-k-)$, occur in two identities. Denote by $\overline{(ijk)(lmn)}$ a determinant product such that in each determinant the natural order occurs. If this arrangement is made in the above schematic products where say $k < l$, and if r numbers lie between k and l, then in the first product they must actually lie between k and l so that in the second product r additional changes of sign are necessary to bring about the natural order. Taking account of the given change of sign and of the fact that $r + 1$ is even or odd according as $k + l$ is even or odd we see that

(78) *The ten determinant products formed from a* P^2_6 *are connected by* 15 *linear relations,*

$$(-1)^k \overline{(ijk)(lmn)} + (-1)^l \overline{(ijl)(kmn)} + (-1)^m \overline{(ijm)(kln)} + (-1)^n \overline{(ijn)(klm)} = 0,$$

where the dash indicates that the natural order is followed in each determinant.

Comparing the relations in (77) and (78) we see that it is permissible to set $\overline{(ijk)(lmn)} = \vartheta^2_{ijk}\vartheta^2_{ijk}(u)$; also for the value $u = 0$ to set $\overline{(ijk)(lmn)} = \vartheta^4_{ijk}$. In the latter case the P^2_6, being expressed in terms of the theta-modular functions of three moduli, must be subject to some condition. If in the theta relation (77) we set $u = P_{in}$ we get

$$(-1)^k \epsilon_{ijk,\,in}\vartheta^2_{ijk}\vartheta^2_{jkn} + (-1)^l \epsilon_{ijl,\,in}\vartheta^2_{ijl}\vartheta^2_{jln} + (-1)^m \epsilon_{ijm,\,in}\vartheta^2_{ijm}\vartheta^2_{jmn} = 0,$$

or

$$\sqrt{\vartheta^4_{ijk}\vartheta^4_{jkn}} + \sqrt{\vartheta^4_{ijl}\vartheta^4_{jln}} + \sqrt{\vartheta^4_{ijm}\vartheta^4_{jmn}} = 0.$$

This implies the following irrational condition on P^2_6:

$$\sqrt{\overline{(ijk)(lmn)}\cdot\overline{(jkn)(lmi)}} + \sqrt{\overline{(ijl)(kmn)}\cdot\overline{(jln)(kmi)}} + \sqrt{\overline{(ijm)(kln)}\cdot\overline{(jmn)(kli)}} = 0.$$

We have seen in § 3 that the condition that P_6^2 be on a conic can be given as

$$\sqrt{(lmn)(kji)\cdot(kjn)(lmi)} + \sqrt{(kmn)(lji)\cdot(ljn)(kmi)} + \sqrt{(kln)(mji)\cdot(mjn)(kli)} = 0.$$

To pass from the first to the second radicand k and l must be interchanged in two products whence from the above the numbers of changes of sign required in the radicands to restore the natural order within the determinants can differ only by even numbers and this condition reduces to the one given above. Hence the existing condition is that P_6^2 be on a conic. To identify the six points on the conic we note that if the conic be taken parametrically the ternary determinant product $(ijk)(lmn)$ becomes the binary determinant product $(ij)(ik)(jk)\cdot(lm)(ln)(mn)$. According to Bolza* this product is $\rho^2\,\vartheta_{ijk}^4$ whence

(79) *If the fourth powers of the ten even thetas for the zero values of the arguments be equated to the ten determinant products formed from* P_6^2, *i. e., if* $\vartheta_{ijk}^4 = \overline{(ijk)(lmn)}$, *they define a* P_6^2 *on a conic whose parameters determine the fundamental binary sextic* (P_6^1) *which is associated with the normal hyperelliptic integrals of the first kind whose canonical periods are the moduli* τ_{11}, τ_{12}, τ_{22} *of the theta functions.*

If the ten products be expressed in terms of $\bar{a}, \cdots, \bar{f}$ as in § 4 we have found that the condition that P_6^2 be on a conic is $a_2^2 - 4a_4 = 0$. This is, in S_4 (variables $\bar{a}, \cdots, \bar{f}$), the quartic spread of Maschke,† the reciprocal of the cubic spread of Richmond,‡ whence we find anew an implicit result due to Maschke:

(80) *The quartic spread,* $a_2^2 - 4a_4 = 0$, *in* S_4 *whose reciprocal is* $\sum_{i=1}^{i=6} u_i^3 = 0$ *where* $\sum_{i=1}^{i=6} u_i = 0$, *can be expressed parametrically and uniformly in terms of the theta-modular functions of genus two.*

Explicit expressions for the coördinates of the P_6^2 in terms of the determinant products and therefore of the ϑ_{ijk}^4's are given below in (89) and (90).

10. The form problems of P_6^1, P_6^2, and P_6^3

For $n = 6$ we have three point sets of which that in S_1 and that in S_3 are associated. If we take the prepared forms of these in the order

---

* Mathematische Annalen, vol. 30 (1887), p. 485; see also the references given there. The relations implied by (79) also are found on pp. 493–4. Indeed the relations (77) are well-known in the theory. They are derived afresh here in order to reconcile the notation for the thetas with that for the points and to identify the signs without comparing lists of particular formulæ.

† Mathematische Annalen, vol. 30 (1887), p. 496; in particular p. 505.

‡ For references see $C2$:

(81)	(1)	$1, 0$	t, s, r, u
	(2)	$0, 1$	$1, 1, 1, 1$
	(3)	$1, 1$	$0, 0, 0, 1$
	(4)	r, u	$0, 0, 1, 0$
	(5)	s, u	$0, 1, 0, 0$
	(6)	t, u	$1, 0, 0, 0$

the Cremona $G_{6!}$ in the space Σ_3 (variables r, s, t, u) for both sets is generated by

(82)		(12)	(23)	(34)	(45)	(56)
	r'	$= ust$	$= u - r$	$= u$	$= s$	$= r$
	s'	$= utr$	$= u - s$	$= s$	$= r$	$= t$
	t'	$= urs$	$= u - t$	$= t$	$= t$	$= s$
	u'	$= rst$	$= u$	$= r$	$= u$	$= u$

The simplest invariant linear system for P_6^1 is obtained from the irrational invariants like $(12)(34)(56)$ all of which contain the normalizing factor u; that for P_6^3 is obtained from irrational invariants like $(3456)(5612)(1234)$. In both cases the linear system is the system of quadrics on the canonical basis in Σ_3. The invariants of P_6^1 have been obtained in § 3; those of P_6^3 have been inferred from the invariants of P_6^1 in $C2$ § 1 where the form problem of $G_{6!}$ in Σ_3 is set forth. This form problem is solved by using the resolvent sextic of Maschke (loc. cit.). If in connection with P_6^1 and P_6^3 we introduce the self-associated P_6^2 Maschke's sextic and its significance appear at once.

Taking P_6^2 in the prepared form

(83)	(1)	$1, 0, 0$	(4)	$1, 1, 1$
	(2)	$0, 1, 0$	(5)	x, y, u
	(3)	$0, 0, 1$	(6)	z, t, u

we have the following equations for the generators of $G_{6!}^2$ in Σ_4 (variables x, y, z, t, u)

(84)		(12)	(23)	(34)	(45)	(56)
	x'	$= y$	$= xt$	$= u - x$	$= uy$	$= z$
	y'	$= x$	$= ut$	$= u - y$	$= ux$	$= t$
	z'	$= t$	$= zy$	$= u - z$	$= zy$	$= x$
	t'	$= z$	$= uy$	$= u - t$	$= tx$	$= y$
	u'	$= u$	$= yt$	$= u$	$= xy$	$= u$

The simplest invariant linear system is obtained from the irrational invariants like $(123)(456)$. If we express these in terms of $\bar{a}, \cdots, \bar{f}$ we find that

$$
(85) \qquad
\begin{aligned}
3\bar{a} &= \rho - 3(ux + ut), & 3\bar{d} &= \rho - 3(uy + uz),\\
3\bar{b} &= \rho - 3(ux + yz), & 3\bar{e} &= \rho - 3(uy + xt),\\
3\bar{c} &= \rho - 3(ut + yz), & 3\bar{f} &= \rho - 3(uz + xt),
\end{aligned}
$$
$$
\rho = u(x + y + z + t) + xt + yz.
$$

Thus the invariant linear system can be expressed in terms of 5 quadrics on the canonical basis in Σ_4, which contain the four lines in the Σ_3, $u = 0$, whose equations are $x = y = 0$, $x = z = 0$, $t = y = 0$, $t = z = 0$. The simplest members of the system are the differences of the six terms in ρ. Four general quadrics of the system having a skew quadrilateral and two further points in common will meet in two additional points $x, \cdots, u$ and $x', \cdots, u'$. Either one of these two points determines the other and the quadrics take proportional values for the two points. This is true of the maps P and P' of two associated P_6^2's whence the two points are corresponding points in the Cremona transformation B of order 5 whose equations are given in (56). The transformation B together with $G_{6!}^2$ determine in Σ_4 a $G_{2\cdot 6!}$ which corresponds in S_2 to the permutations of the points of the associated P_6^2 and Q_6^2 and to the interchange of the sets. The element B is permutable with each element of $G_{6!}^2$ and $G_{2\cdot 6!}$ is the product of the groups $G_{6!}$ and the G_2 with elements B and $B^2 = 1$. The invariants of $G_{6!}$ and $G_{2\cdot 6!}$ have been obtained in § 5.

The form problem of $G_{6!}^2$ is as follows:

(86) *Given the values of* a_2, a_3, a_4, a_5, a_6, *and* $d_2\sqrt{d}$, *subject to the syzygy which connects* $(d_2\sqrt{d})^2$ *with* $a_2, \cdots, a_6$ *to find the* 720 *points in* Σ_4 *for which these invariants take the assigned values.*

Here

$$
d_2 = \sqrt{a_2^2 - 4a_4} = \begin{vmatrix} (341)(561) & (531)(461)\\ (342)(562) & (532)(462)\end{vmatrix}
$$
$$
= u\,[yz(x + t - u) - xt(y + z - u)];
$$

while $d = \prod(\bar{d} - \bar{a})^2 = \prod(12, 34, 56)^2 = \prod(u\,[x + t - z - y])^2$. If $d_2 = 0$, P_6^2 is on a conic; if $d = 0$, P_6^2 is on three lines which meet in a point. Let us first verify that the given values determine 720 points. From a_2, $\cdots$, a_6 there is derived, by using the permutations of the roots of the sextic with these coefficients, 720 value systems $\bar{a}, \cdots, \bar{f}$. To each of these systems there correspond two points. Under the even subgroup $G_{\frac{1}{2}6!}$ of $G_{6!}$, which is invariant under $G_{2\cdot 6!}$ these 1,440 points divide into four conjugate sets of 360 each, say S_0, S_1, S_0', S_1'. If π is an odd permutation of $G_{6!}$ the elements of $G_{2\cdot 6!}$ divide with regard to $G_{\frac{1}{2}6!}$ into four partitions, namely, $G_{\frac{1}{2}6!}$, $G_{\frac{1}{2}6!}\,\pi$, $G_{\frac{1}{2}6!}\,B$, $G_{\frac{1}{2}6!}\,\pi B$. These partitions permute the four sets above as follows: $(S_0)(S_1)(S_0')(S_1')$, $(S_0 S_1)(S_0' S_1')$, $(S_0 S_0')(S_1 S_1')$, $(S_0 S_1')(S_1 S_0')$. The element π effects not only a change of sign in $\sqrt{d}$ but also a change of sign in d_2. The element B effects a change of sign in d_2 only. Thus only the two sets S_0, S_1 arise from the given values while for the sets S_0', S_1' the invariant $d_2\sqrt{d}$ takes the opposite sign. We have already remarked (§ 1) that the sets P_6^2 and Q_6^2 can coincide in two ways if only one condition be imposed. They can coincide in the identical order if $d_2 = 0$, and in an odd order like $(12)(34)(56)$ if $\bar{a} = \bar{d}$. Thus as we should expect S_0', S_1' coincide with S_0, S_1 if either $d_2 = 0$ or $d = 0$.

To solve the given form problem we adjoin $\sqrt{d}$ for the known sextic. Then both d_2 and $\sqrt{d}$ are rationally known, and the group of the problem is reduced to $G_{\frac{1}{2}6!}$. From one of the resulting 360 value systems of $\bar{a}, \cdots, \bar{f}$ we get from (85)

$$
(87)\qquad
\begin{aligned}
6yz &= \rho + 3(\bar{a} - \bar{b} - \bar{c}), & 6xt &= \rho + 3(\bar{d} - \bar{e} - \bar{f}),\\
6ut &= \rho + 3(-\bar{a} + \bar{b} - \bar{c}), & 6uz &= \rho + 3(-\bar{d} + \bar{e} - \bar{f}),\\
6ux &= \rho + 3(-\bar{a} - \bar{b} + \bar{c}), & 6uy &= \rho + 3(-\bar{d} - \bar{e} + \bar{f}).
\end{aligned}
$$

The product of the left-hand members in each of the two columns are equal. This fact applied to the right-hand members yields a quadratic equation in ρ. If we set

$$
\begin{aligned}
\sigma_1 &= \bar{a} + \bar{b} + \bar{c}, & \tau_1 &= \bar{d} + \bar{e} + \bar{f},\\
\sigma_2 &= \bar{b}\bar{c} + \bar{c}\bar{a} + \bar{a}\bar{b}, & \tau_2 &= \bar{e}\bar{f} + \bar{f}\bar{d} + \bar{d}\bar{e},\\
\sigma_3 &= \bar{a}\bar{b}\bar{c}, & \tau_3 &= \bar{d}\bar{e}\bar{f},
\end{aligned}
$$

whence

$$a_2 = \sigma_2 + \tau_2 + \sigma_1\tau_1, \qquad a_4 = \sigma_3\tau_1 + \sigma_1\tau_3 + \sigma_2\tau_2,$$

this quadratic equation is

$$\rho^2(\sigma_1 - \tau_1) - 12\rho(\sigma_2 - \tau_2) + 9[(\sigma_1 - \tau_1)\sigma_1\tau_1 + 4(\sigma_1\sigma_2 - \tau_1\tau_2) - 8(\sigma_3 - \tau_3)] = 0.$$

Its discriminant easily reduces on using $\sigma_1 + \tau_1 = 0$ to $36(a_2^2 - 4a_4) = 36d_2^2$. Hence

$$(88)\qquad \rho = 6\frac{(\sigma_2 - \tau_2) + d_2}{(\sigma_1 - \tau_1)}.$$

Since d_2 and therefore ρ is now rationally known we have

$$
(89)\qquad
\begin{aligned}
6u\cdot x &= \rho + 3(-\bar{a} - \bar{b} + \bar{c}),\\
6u\cdot y &= \rho + 3(-\bar{d} - \bar{e} + \bar{f}),\\
6u\cdot z &= \rho + 3(-\bar{d} + \bar{e} - \bar{f}),\\
6u\cdot t &= \rho + 3(-\bar{a} + \bar{b} - \bar{c}),\\
6u\cdot u &= \frac{6uz\cdot 6uy}{6yz} = \frac{[\rho + 3(-\bar{d} + \bar{e} - \bar{f})][\rho + 3(-\bar{d} - \bar{e} + \bar{f})]}{[\rho + 3(\bar{a} - \bar{b} - \bar{c})]}.
\end{aligned}
$$

This determines the point x, y, z, t, u in Σ_4 which is a solution of (86). The

720 points can be obtained from the one either by effecting the operations of $G_{6!}$ or by permuting $\bar{a}, \cdots, \bar{f}$ in (89) provided the sign of d_2 be changed for an odd permutation. Another expression for the coördinates can be obtained from (89) which introduces the coefficients of the tritangent planes of C^3 as given in (47). This is

$$
(90)\qquad
\begin{aligned}
\lambda x &= d_2 - \overline{ab}, \qquad \lambda y = d_2 + \overline{de},\\
\lambda z &= d_2 + \overline{df}, \qquad \lambda t = d_2 - \overline{ac},\\
\lambda u &= \frac{(d_2 + \overline{de})(d_2 + \overline{df})}{(d_2 - \overline{bc})}, \qquad \lambda = (\sigma_1 - \tau_1)u.
\end{aligned}
$$

The Cremona $G_{6!}$ furnishes an expression of the algebraic relations which must exist among these coefficients.

The above solution of the form problem of the general P_6^2 is based on the solution of the sextic. On the other hand the form problem of the particular self-associated P_6^2 is of vital importance in the solution of the sextic itself. The sextic S defines a P_6^1. After the adjunction of $\sqrt{D}$, S has a rational sextic resolvent Σ, the sextic of Joubert, which is suggested at once by P_6^3. These two point sets—the one on a line and the other on a twisted cubic—are evidently related to the P_6^2 on a conic. And indeed this P_6^2 determines the sextic Φ of Maschke whose roots can be identified with $\bar{a}, \cdots, \bar{f}$. According to $C2$, p. 323 (32), the roots of Σ can be rationally expressed in terms of the roots of Φ and in turn the roots of S in terms of the roots of Σ. Moreover the solution of Φ in terms of hyperelliptic modular functions is apparent from the last paragraph. Thus the solution of the general sextic S breaks into two parts.

The first or transcendental part of the solution consists of the following operations: From the periods of a pair of normal hyperelliptic integrals of the first kind attached to the Riemann surface $y^2 = S$, the moduli τ_{11}, τ_{12}, τ_{22} of the theta functions are obtained. Then from the well-known series which define them the values of the ten ϑ_{ijk}^4's (or of any five linearly independent ones) are calculated.

The second or algebraic part is as follows: From the ten ϑ_{ijk}^4's the determinant products $(ijk)(lmn)$ of the P_6^2 are obtained in (79) according to which this P_6^2 is parametrically projective to S. After finding $\bar{a}, \cdots, \bar{f}$ from the $(ijk)(lmn)$, either we can use (89) or (90) to determine the P_6^2 and can find the parameters of its six points on the conic, or having the roots of Σ we can get from $C2$, p. 319 (19), a solution of the form problem of P_6^3. In either case we have the roots of a sextic projective to S and can obtain therefrom, by using the transformation, $C2$, p. 318 (17), the roots of S itself. It is worth noting that in this algebraic part no accessory irrationalities are required.

BALTIMORE,
December 1, 1914.